First midterm

Dave Bayer, Modern Algebra, February 25, 1998

All rings are commutative with identity, as in our text. Recall that an *integral domain* R is a nonzero ring having no zero divisors. In other words, if $ab = 0$ then $a = 0$ or $b = 0$.

[1] Consider the ideal $I = (2, x^3 + x) \subset \mathbb{Z}[x]$.

(a) Describe the ideal I in words, listing enough elements of I to make the pattern clear. How would you tell at a glance if a given polynomial $f(x) \in \mathbb{Z}[x]$ belongs to I?

(b) List representatives for the elements of the quotient ring $R = \mathbb{Z}[x]/I$, and describe the multiplication rule in R for these representatives.

(c) Is R an integral domain?

[2] Let $a = 3 - i$ and $b = 2i$ be elements of the Gaussian integers $\mathbb{Z}[i]$.

(a) Find q_1, $c \in \mathbb{Z}[i]$ so $a = q_1 b + c$ with $|c| < |b|$.

(b) Now find q_2, $d \in \mathbb{Z}[i]$ so $b = q_2 c + d$ with $|d| < |c|$.

(c) Express $(a, b) \subset \mathbb{Z}[i]$ as a principal ideal.

[3] Prove that the following two statements are equivalent, for a nonzero ring R and elements a, b, $c \in R$:

(a) R is an integral domain.

(b) We can *cancel* in R: If $ab = ac$ for $a \neq 0$, then $b = c$.

[4] Prove that if an integral domain R has only finitely many elements, then R is a field.

[5] Let $\mathbb{F}_5$ be the finite field with 5 elements. Find a polynomial $f(x)$ in the polynomial ring $\mathbb{F}_5[x]$ such that the quotient ring $\mathbb{F}_5[x]/(f(x))$ is a field with 25 elements.