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0 1 2 3 4 5 6

S16 Exam 2 Problem 1
Combinatorics, Dave Bayer

test1a2p1

Test 1

Name solutions Uni ______

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[1] Give a proof of Burnside's Lemma: If a group G acts on a set of patterns X, then the number of distinct patterns up to symmetry is equal to the average number of patterns fixed by an element of the group:

$$\frac{1}{|G|}\sum_{g\in G}|X^g|$$

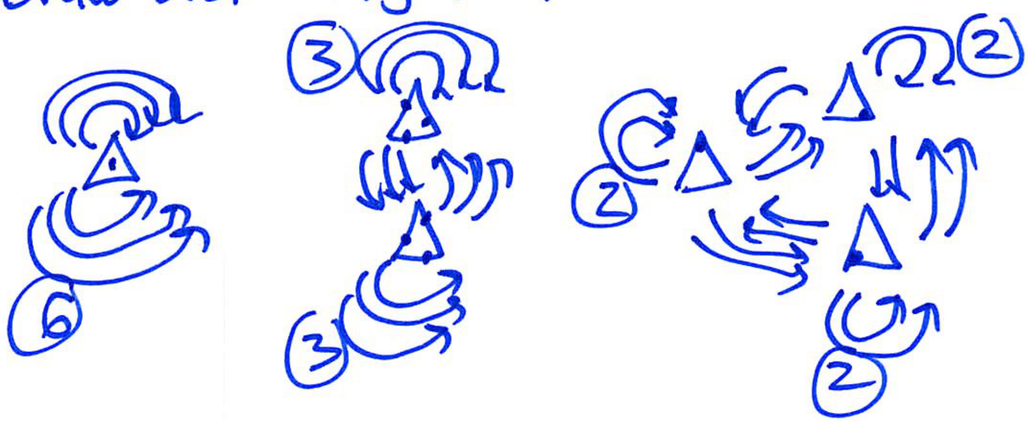
$$\parallel$$

$$\frac{1}{|G|}\left|\{(g,x) \mid gx = x\}\right|$$

count self-loops (pattern carried to self by element of g) and divide by $|G|$

Everything divides evenly in group theory. ← (assume this, till you take Modern Algebra)

Draw orbit diagram for action of G on X



Every orbit has a total of $|G|$ self loops

⇒ # patterns up to symmetry

= # orbits

= # self-loops ÷ $|G|$

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○ ○ ○ ○ ○ ○ ◉
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S16 Exam 2 Problem 2
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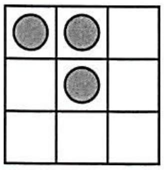
test1a2p2

Test 1

[2] How many ways can three checkers be placed on a 3×3 checkerboard, up to symmetry? Consider both rotations and flips.



Id

·	·	·
·	·	·
·	·	·

↻↺

B	C	B
C	A	C
B	C	B

↻

B	C	D
E	A	E
D	C	B

↔ (↕)

D	A	D
E	B	E
F	C	F

⤢ (⤡)

A	D	E
D	B	F
E	F	C

$|G| = 8$

g	#	$\lvert X^g \rvert$	Σ
Id	1	84	84
↻↺	2	0	0
↻	1	4	4
↔ ↕	2	10	40
⤡ ⤢	2		
8			$128/8 = \boxed{16}$

Id $\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 12 \cdot 7 = 70 + 14 = 84$

↻ A, one of B, C, D, E   4

flips (one of A, B, C)(one of D, E, F)

$3 \cdot 3 = 9$

or   D, E and F   1

✓

[2]

Check:

Middle:

	•	

•	A	B
	•	C
		D

4

X	•	X
	•	A
X	B	X

2

6

No middle:

3 corners

•		•
	X	
		•

1

2 corners adjacent

•	A	•
B	X	
X	C	X

3

2 corners opposite

•	A	X
	X	
X		•

1

5

1 corner

•		X
	X	
X		X

•	•	
•		

1

•	•	X
X	X	A
X	B	X

2

•		
		•
	•	

1

No corners

X	•	X
	X	•
X	•	X

1

5

$\boxed{16}$ ✓

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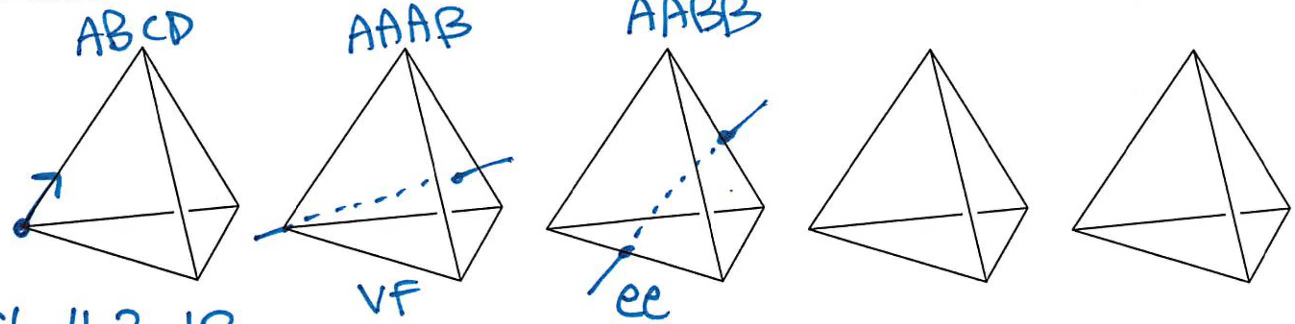
0 1 2 3 4 5 6

S16 Exam 2 Problem 3
Combinatorics, Dave Bayer

test1a2p3

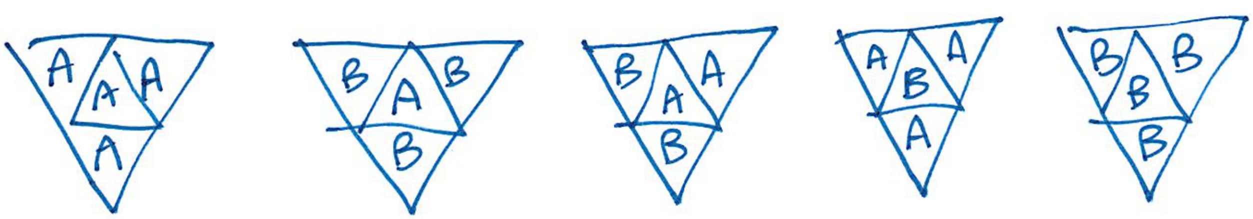
Test 1

[3] Up to rotational symmetry, how many ways can we color the four faces of a tetrahedron, using at most two colors?



$|G| = 4 \cdot 3 = 12$

g	$\#$	$\lvert X^g \rvert$	Σ
Id	1	2^4	16
$\frac{1}{3}\circlearrowright$ VF	8	2^2	32
$\frac{1}{2}\circlearrowright$ ee	3	2^2	12
	12		$60/12 = \boxed{5}$



(tetrahedrons as bearskin rugs) (A, B are colors)

$\boxed{5}$ ✓

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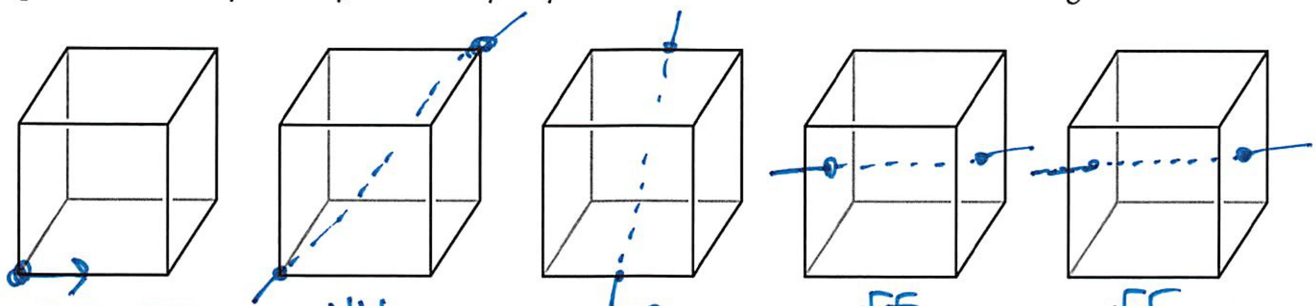
0 1 2 3 4 5 6

S16 Exam 2 Problem 4
Combinatorics, Dave Bayer

test1a2p4

Test 1

[4] Up to rotational symmetry, how many ways can we choose three of the twelve edges of a cube?



$|G| = 8 \cdot 3 = 24$ VV ee FF $\frac{1}{4}\Omega$ ff $\frac{1}{2}\Omega$

	g	#	$\lvert X^g \rvert$	Σ
	Id	1	220	220
$\frac{1}{3}\Omega$	VV	8	4	32
$\frac{1}{2}\Omega$	ee	6	10	60
$\frac{1}{4}$	ff	6	0	
$\frac{1}{2}$	ff	3	0	
		24		312

$312 = 240 + 72$

$\div 24 \quad 10 + 3 = \boxed{13}$

$\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 2 \cdot 11 \cdot 10 = 220$

VV $3, 3, 3, 3$ choose 1

ee $\underbrace{1, 1}_{\text{choose } 1}, \underbrace{2, 2, 2, 2, 2}_{\text{choose } 1}$ $\quad 2 \cdot 5 = 10$

ff $\frac{1}{4}$ $4, 4, 4$ zero

ff $\frac{1}{2}$ $2, 2, 2, 2, 2, 2$ zero

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0 1 2 3 4 5 6

S16 Exam 2 Problem 5
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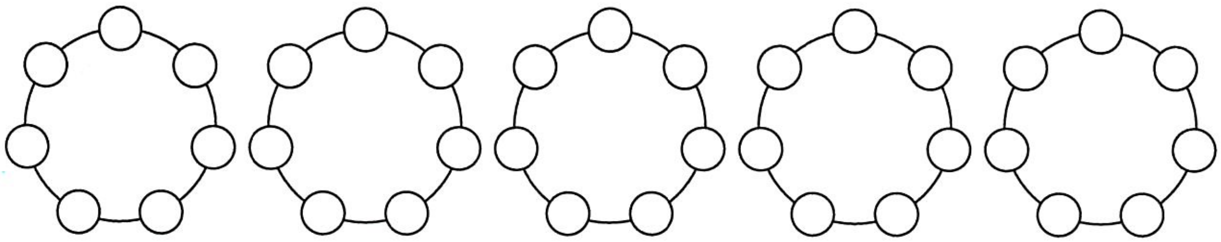
test1a2p5

Test 1

[5] Up to symmetry, how many ways can the beads of a seven bead necklace be colored, using exactly three colors? Consider only rotations, and use all three colors.



$|G| = 7$

g	#	$\lvert X^g \rvert$	Σ
Id	1	k^7	k^7
turn	6	k	$6k$

k colors (at most)

k	k^7+6k	$\div 7$
1	7	1
2	140	20
3	2205	315

$3^7 = 3^4 \cdot 3^3 = 81 \cdot 27$

$6 \cdot 3 = 18$

$$\begin{array}{r} 27 \\ 81 \\ \hline 27 \\ 216 \\ \hline 2187 \\ 18 \\ \hline 2205 \end{array}$$

$2100 + 105$

$\div 7 \quad 300 + 15$

Inclusion/Exclusion

at most	ABC	315
− at most	AB	−20
	AC	−20
	BC	−20
+ at most	A	1
	B	1
	C	1

$\underbrace{315 - 60}_{255} + 3 = \boxed{258}$

✓

[5] check:

Every pattern using all 3 colors has an orbit of size 7.

So, # raw patterns ÷ 7

$$
\begin{array}{lrl}
\text{at most 3:} & 3^7 = 2187 & \\
-\text{at most 2:} & 2^7 = -128 & \\
 & -128 & \\
 & -128 & \\
+\text{at most 1:} & +3 &
\end{array}
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\} \; 3 \cdot 127 = 381
$$

$$
\begin{array}{r}
2187 \\
-381 \\
\hline
1806
\end{array}
\quad \div 7
$$

$$
\begin{array}{r|r}
1400 & 200 \\
350 & 50 \\
56 & 8 \\
\hline
 & \boxed{258}
\end{array}
$$

✓