[1] Up to rotational symmetry, how many nine bead necklaces have three red beads and six blue beads?

\[
\frac{1}{16!} \sum_{g \in G} |x^g| \quad \text{where} \quad G = \text{9 rotations} \\
X = \text{all raw patterns} \\
\left(\frac{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84\right)
\]

\[
\begin{array}{c|c|c}
\text{k/9 turn} & |x^g| & \text{choice one group red} \\
\hline
0 & 84 & ABC \\
1 & 0 & \uparrow \\
2 & 0 & \leftarrow \\
3 & 3 & \text{two groups blue} \\
4 & 0 & \text{ABC} \\
5 & 0 & \text{CA} \\
6 & 0 & \text{CB} \\
7 & 0 & \text{AC} \\
8 & 0 & \text{BA} \\
90 \div 9 = 10 & \\
\end{array}
\]

check: ticks on inside, outside of five example necklaces above show positions of red beads for all 10 possibilities.
[2] Up to symmetry (rotations and flips), how many ways can the squares of a 4 by 4 checkerboard be colored using $n$ colors?

$$G = \{(1, 1, 2, 0, 0), (0, 1, 2, 1, 1), (1, 2, 0, 0, 1)\} \quad |G| = 3$$

$$\frac{n^6 + 2n^4 + 3n^8 + 2n^4}{8}$$

basic check: $n=1$ \[\frac{1+2+3+2}{8} = 1 \checkmark\]
[3] Up to rotational symmetry, how many ways can the eight corners of a cube be colored using $n$ colors?

$\frac{n^8 + 17n^4 + 6n^2}{24}$

basic check: $n=1 \quad \frac{1+17+6}{24} = 1$ \(\Box\)
[4] Give a proof of Burnside's Lemma: If a group $G$ acts on a set of patterns $X$, then the number of distinct patterns up to symmetry is equal to the average number of patterns fixed by an element of the group:

$$\frac{1}{|G|} \sum_{g \in G} |X^g|$$

Going through set of raw patterns, we want to count each distinct pattern up to symmetry a total of once. So each time we see a raw pattern we want to add $\frac{1}{N}$ if the pattern will appear $N$ times.

$|G|$ = total number of symmetries
$|G_x|$ = number of symmetries that fix a pattern $x$

$N = \frac{|G|}{|G_x|}$ because effects of $g \in G$ break up into equal sized subsets (the same number of symmetries take $x$ to $y$ as fix $x$.)

so initial formula is $\sum_{x \in X} \frac{|G_x|}{|G|} = \frac{1}{|G|} \sum_{x \in X} |G_x|$

Now $\sum_{x \in X} |G_x| = \left| \left\{ (g,x) \mid g \cdot x = x \right\} \right| = \sum_{g \in G} |X^g|$ so

final formula is $\left[ \frac{1}{|G|} \sum_{g \in G} |X^g| \right]$
[5] Up to symmetry (rotations and flips), how many ways can one mark six out of the 24 triangles of the following figure?

\[ |G| = 12 \quad 6 \text{ rotations, 6 flips} \]

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}, \quad 3 \text{ side flips}, 3 \text{ corner flips} \]

1: 24 triangles, 24 = 24:1, each triangle stands alone

\[ 1 \times 3! = \left( \frac{24}{6} \right) \]

1/2: 24 = 12:2, must mark 3 pairs out of 12 pairs, \( \binom{12}{3} \)

1 \ 1/3, 2/3 24 = 8:3 must mark 2 triples out of 8 \( \binom{8}{2} \)

1 \ 1/6, 5/6 24 = 4:6 must mark one set of 6, out of 4 \( \binom{4}{1} \)

Side Flips: 4 triangles on axis, 10 pairs of triangles off axis

\[ 6 = 1 + 1 + 1 + 2 \quad \left( \frac{4}{4} \right) \left( \frac{10}{10} \right) \]

\[ 6 = 1 + 1 + 2 + 2 \quad \left( \frac{4}{4} \right) \left( \frac{10}{2} \right) \]

\[ 6 = 2 + 2 + 2 \quad \left( \frac{4}{4} \right) \left( \frac{6}{6} \right) \]

Corner Flips: 12 pairs off axis \( \binom{12}{3} \)

\[ \frac{1}{12} \left[ \left( \frac{24}{6} \right) \left( \frac{12}{3} \right) + 2 \left( \frac{8}{2} \right) + 2 \left( \frac{4}{1} \right) + 3 \left( \frac{4}{4} \right) \left( \frac{10}{1} \right) + \left( \frac{4}{2} \right) \left( \frac{10}{2} \right) + \left( \frac{4}{3} \right) \left( \frac{10}{3} \right) \right] + 3 \left( \frac{12}{3} \right) \]