S15 Exam 2 Problem 1
Combinatorics, Dave Bayer

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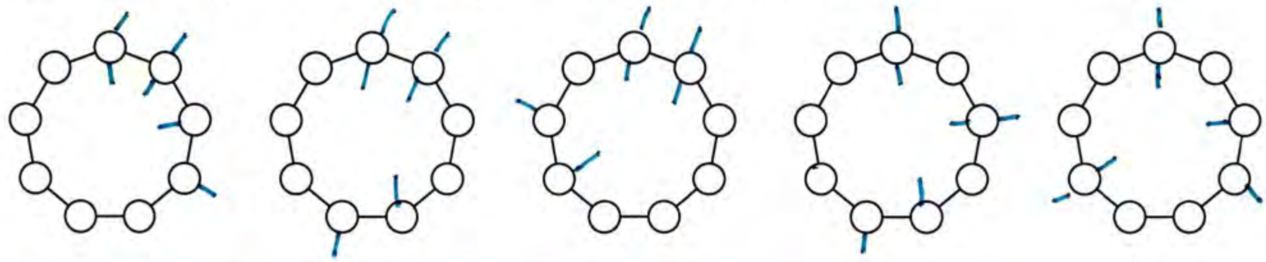
Exam 01

Name solutions Uni

[1] Up to rotational symmetry, how many nine bead necklaces have three red beads and six blue beads?



$\frac{1}{|G|}\sum_{g\in G}|X^g|$ where G = 9 rotations
X = all raw patterns $\binom{9}{3} = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = 3\cdot 4\cdot 7 = 84$

k/9 turn	$\lvert X^g\rvert$
0	84
1	0
2	0
3	3
4	0
5	0
6	3
7	0
8	0

B C A B
A C
C B A

choose one group red
two groups blue

$90 \div 9 = \boxed{10}$

check: ticks on inside, outside of five example necklaces above show positions of red beads for all 10 possibilities.

S15 Exam 2 Problem 2
Combinatorics, Dave Bayer

exam01a2p2

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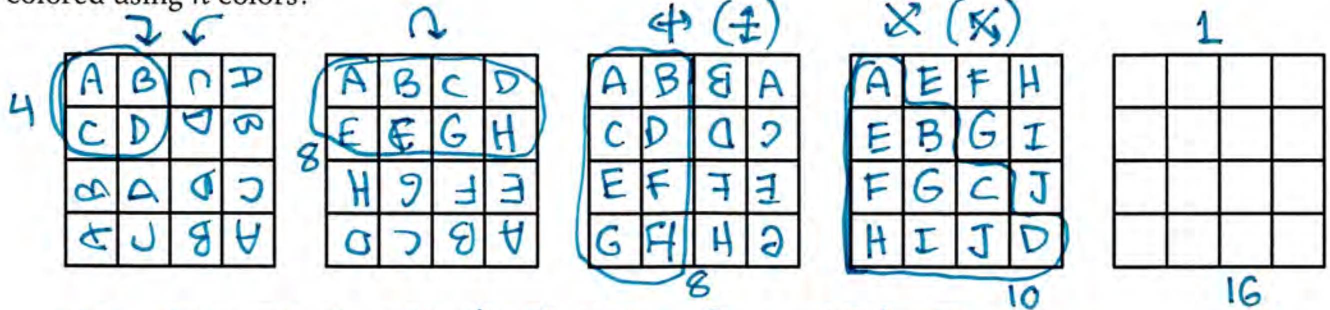
Exam 01

[2] Up to symmetry (rotations and flips), how many ways can the squares of a 4 by 4 checkerboard be colored using n colors?



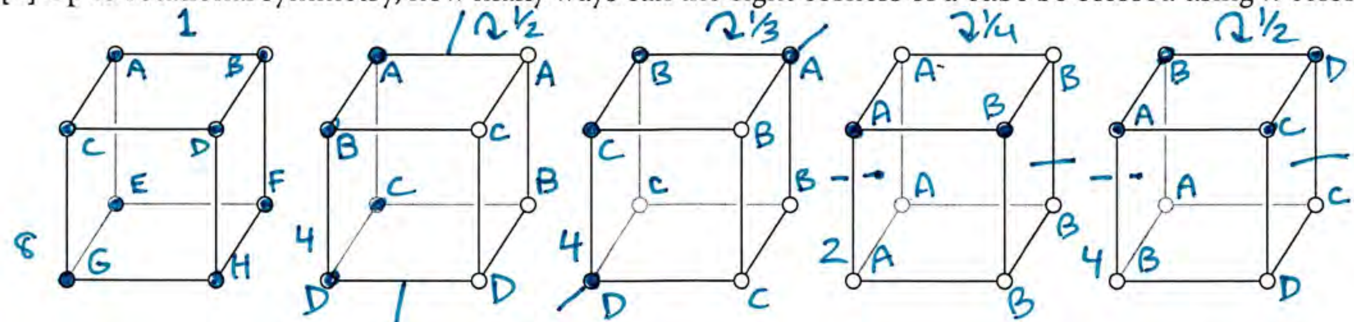
$G = \{1, \circlearrowright, \curvearrowright, \circlearrowleft, \leftrightarrow, \updownarrow, \nwarrow\!\searrow, \nearrow\!\swarrow\}$ $\quad |G| = 8$

$$\boxed{\frac{n^{16} + 2n^{10} + 3n^{8} + 2n^{4}}{8}}$$

basic check: $n=1$ $\quad \dfrac{1+2+3+2}{8} = 1$ ✓

Exam 01

[3] Up to rotational symmetry, how many ways can the eight corners of a cube be colored using n colors?



24 rotational symmetries

turn	0	½, edge	⅓, corner	¼, side	½, side
#g	1	6	8	6	3
$\lvert X^g \rvert$	n^8	n^4	n^4	n^2	n^4

$$\boxed{\frac{n^8 + 17n^4 + 6n^2}{24}}$$

basic check: $n=1$ $\quad \frac{1+17+6}{24} = 1$ ✓

Exam 01

[4] Give a proof of Burnside's Lemma: If a group G acts on a set of patterns X, then the number of distinct patterns up to symmetry is equal to the average number of patterns fixed by an element of the group:

$$\frac{1}{|G|}\sum_{g\in G}|X^g|$$

Going through set of raw patterns, we want to count each distinct pattern up to symmetry a total of once. So each time we see a raw pattern we want to add $\frac{1}{N}$ if the pattern will appear N times.

$|G|$ = total number of symmetries
$|G_x|$ = number of symmetries that fix a pattern x
$N = |G|/|G_x|$ because effects of $g \in G$ break up into equal sized subsets (the same number of symmetries take x to y as fix x.)

so initial formula is $\displaystyle\sum_{x\in X}\frac{|G_x|}{|G|} = \frac{1}{|G|}\sum_{x\in X}|G_x|$

Now $\displaystyle\sum_{x\in X}|G_x| = \left|\{(g,x) \mid gx = x\}\right| = \sum_{g\in G}|X^g|$ so

final formula is $\boxed{\displaystyle\frac{1}{|G|}\sum_{g\in G}|X^g|}$ //

S15 Exam 2 Problem 5
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exam02a2p5

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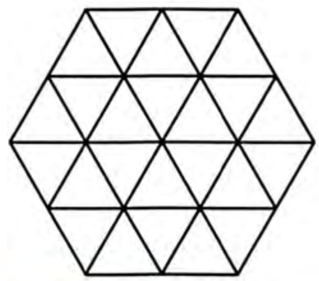
Exam 02

[5] Up to symmetry (rotations and flips), how many ways can one mark six out of the 24 triangles of the following figure?



$|G| = 12$ 6 rotations, 6 flips

$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{6},\ \tfrac{5}{6},$ 3 side flips, 3 corner flips

1: 24 triangles, $24 = 24 \cdot 1$, each triangle stands alone

$|X^g| = \binom{24}{6}$

$\tfrac{1}{2}$: $24 = 12 \cdot 2$, must mark 3 pairs out of 12 pairs, $\binom{12}{3}$

2 $\tfrac{1}{3}, \tfrac{2}{3}$ $24 = 8 \cdot 3$ must mark 2 triples out of 8 $\binom{8}{2}$

2 $\tfrac{1}{6}, \tfrac{5}{6}$ $24 = 4 \cdot 6$ must mark one set of 6, out of 4 $\binom{4}{1}$

side flips: 4 triangles on axis, 10 pairs of triangles off axis

$6 = 1+1+1+1+2$ $\binom{4}{4}\binom{10}{1}$

$6 = 1+1+2+2$ $\binom{4}{2}\binom{10}{2}$

$6 = 2+2+2$ $\binom{4}{0}\binom{10}{3}$

corner flips: 12 pairs off axis $\binom{12}{3}$

$$\frac{1}{12}\left[\binom{24}{6} + \binom{12}{3} + 2\binom{8}{2} + 2\binom{4}{1} + 3\left[\binom{4}{4}\binom{10}{1} + \binom{4}{2}\binom{10}{2} + \binom{4}{0}\binom{10}{3}\right] + 3\binom{12}{3}\right]$$