

Exam 1

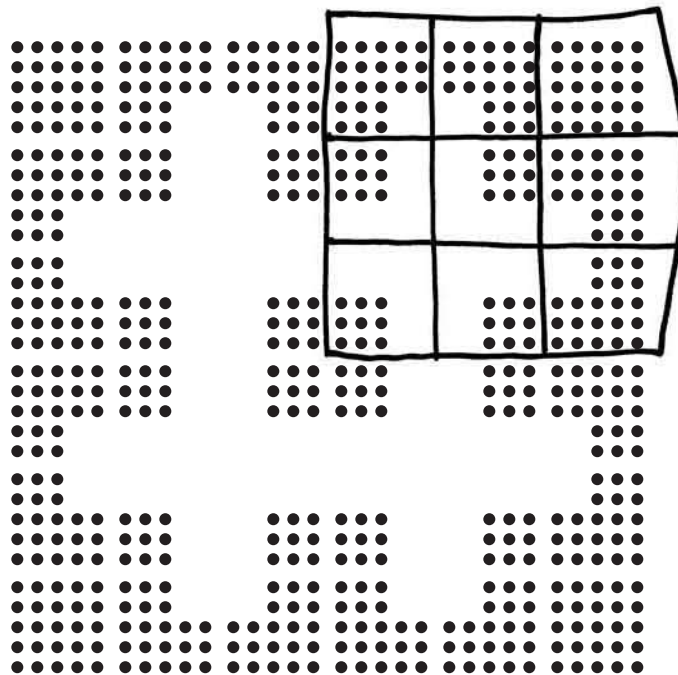
Combinatorics, Dave Bayer, February 11, 2014

Name: Solutions Uni: _____

[1]	[2]	[3]	[4]	[5]	Total

If you need more than one page for a problem, clearly indicate on each page where to look next for your work.

[1] How many dots?



$$30 \times 30 - 4(4 \cdot 24) + 4(4 \cdot 4)$$

$$= 900 - 4 \cdot 4 \cdot 20 = 900 - 320 = \boxed{580}$$

or $\underbrace{15 \cdot 15 - 2(4 \cdot 12) + 16}_{\text{count a quarter}} = 225 - 96 + 16 = 145$
 $\times 4 = 580 \checkmark$

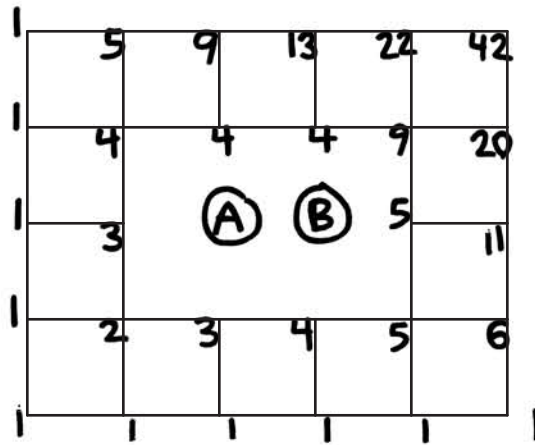
or

21	21	25
9	9	21
9	9	21

$$4 \cdot 21 + 4 \cdot 9 + 25$$

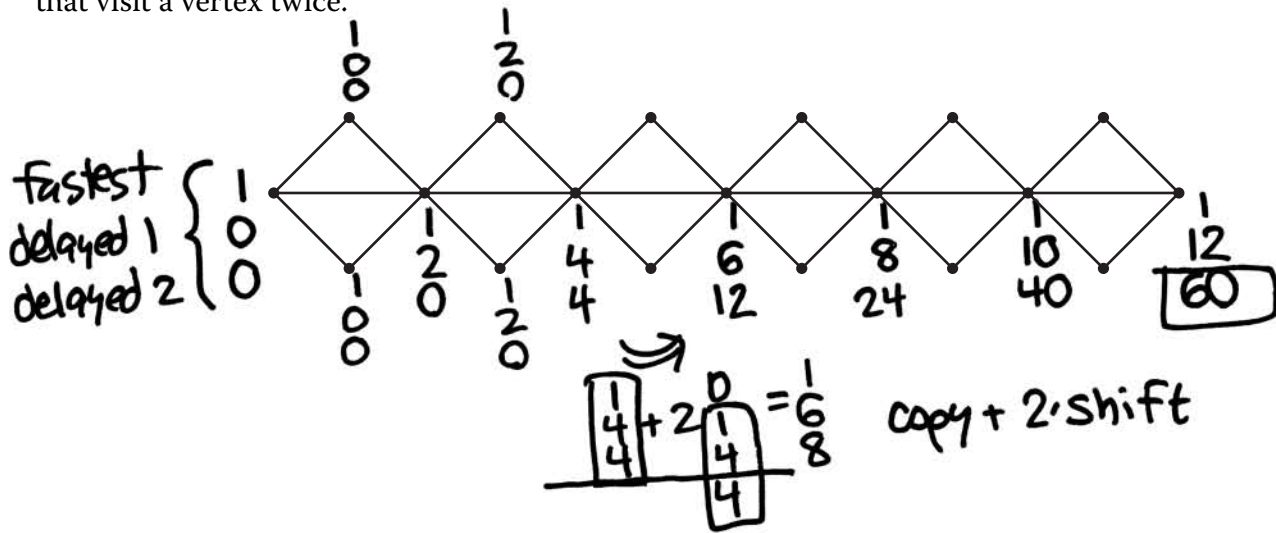
$$= 4 \cdot 30 + 25 = 145 \times 4 = 580 \checkmark$$

[2] How many paths are there from the lower left corner to the upper right corner of this grid, moving only up or to the right?



$$\begin{aligned}
 & \overbrace{\binom{9}{4}}^{\text{all}} - \overbrace{\binom{4}{2}\binom{5}{2}}^{(A)} - \overbrace{\binom{5}{2}\binom{4}{2}}^{(B)} + \overbrace{\binom{4}{2}\binom{4}{2}}^{(A),(B)} \\
 & \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} - \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} - \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} + \frac{4 \cdot 3}{2 \cdot 1} \cdot \frac{4 \cdot 3}{2 \cdot 1} \\
 & 126 - 6 \cdot 10 - 10 \cdot 6 + 6 \cdot 6 \\
 & 126 - 120 + 36 = \boxed{42}
 \end{aligned}$$

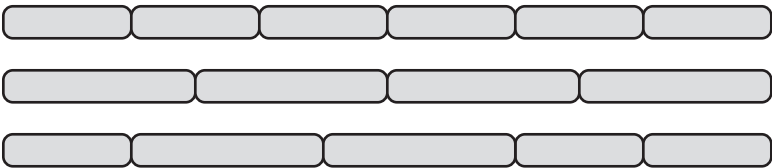
[3] How many paths are there of length 8 from the leftmost vertex to the rightmost vertex? Ignore paths that visit a vertex twice.



Take long way around 2 of 6 diamonds
choose upper or lower each time

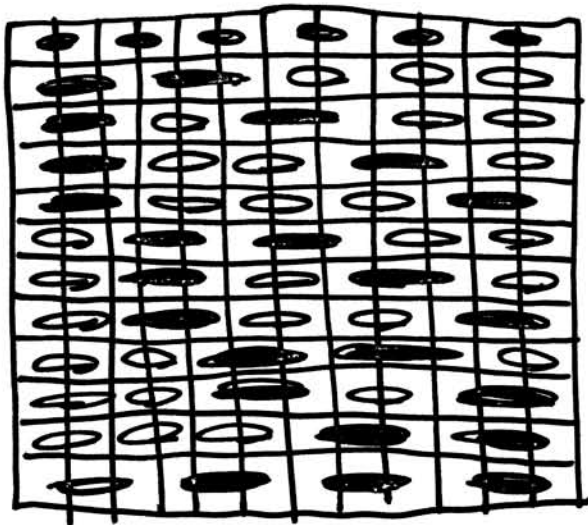
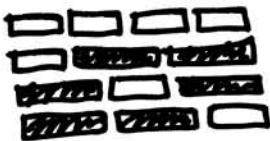
$$\binom{6}{2} \cdot 2^2 = 15 \cdot 4 = \boxed{60}$$

[4] How many ways can one fill a tube of length 12, using sticks of length 2 and 3? Three of the possibilities are shown below:

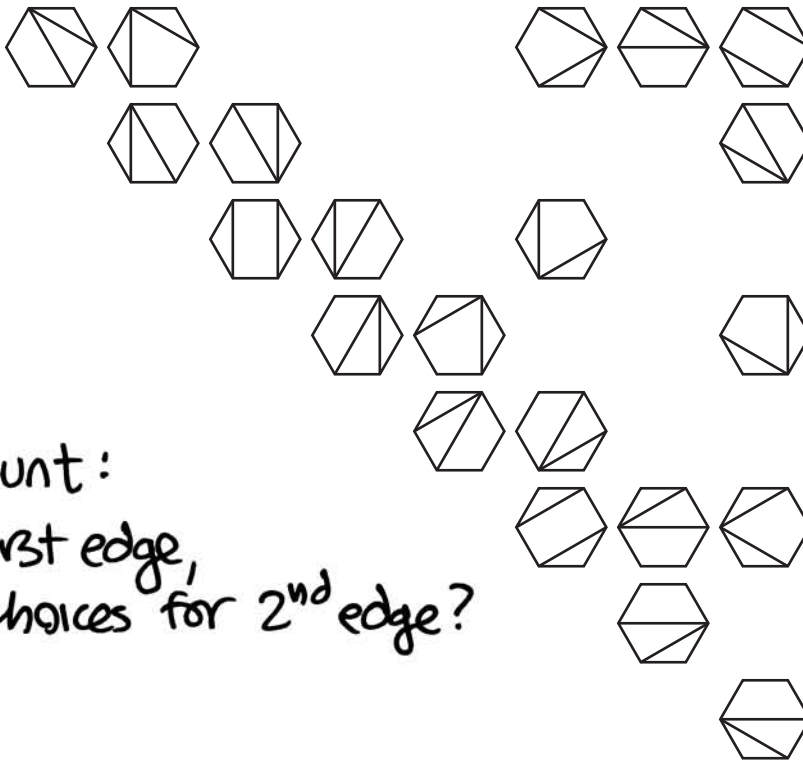


tube length	ways
0	1
1	0
2	1
3	1
4	1
5	2
6	2
7	3
8	4
9	5
10	7
11	9
12	12

is first stick a 2 or 3?



[5] There are 21 ways to cut a hexagon into three pieces. How many ways are there to cut an octagon into three pieces?



Double-count:

For each first edge,
how many choices for 2nd edge?

one cut

4-gon	2
5	$5 = \binom{5}{2} - 5$
6	$9 = \binom{6}{2} - 6$
7	$14 = \binom{7}{2} - 7$
8	$20 = \binom{8}{2} - 8$

$$6 \text{ (pentagons)} + 3 \text{ (hexagons)} = \frac{6 \cdot 5 + 3(2+2)}{2} = 21 \quad \checkmark$$

$$8 \text{ (heptagons)} + 8 \text{ (hexagons)} + 4 \text{ (pentagons)} \\ \frac{[8 \cdot 14 + 8(9+2) + 4(5+5)]}{2} \\ = 240/2 = \boxed{120}$$

check: Count map



$\rightarrow 4/5$



$\rightarrow 4/5$

3 36	3 45	3 54	3 63	4 35	4 44	4 53	5 34	5 43	6 33
3 36	3 45	3 54	3 63	4 35	4 44	4 53	5 34	5 43	6 33
1 4	1 6	1 8	1 10	3 6	3 9	3 12	6 8	6 12	10 10
32				36			32		20

$$32 + 36 + 32 + 20 = 120 \quad \checkmark$$