Practice Exam 2

Modern Algebra I, Dave Bayer, March 18, 2010

The following problems are representative of topics and problems that may appear on our second exam.

[1] Let G be a finite group acting on the set X. Show that for each $x \in X$, the order $|O_x|$ of the orbit O_x divides the order $|G|$ of the group G.

(The orbit of x can also be denoted by Gx, not to be confused with the stabilizer G_x of x.)

[2] Let G be a finite group acting on the set X, and let $x, y \in X$. Define the *stabilizer* G_x of x. Prove that if x and y are in the same orbit, then G_y is conjugate to G_x.

[3] There are $3^4 = 81$ ways of labeling the four corners of a square with integers from the set $\{1, 2, 3\}$. Let X be this set of labeled squares, and let the dihedral group D_4 act on X by rotations and flips of the square. Describe the decomposition of X into orbits.

[4] Let G be a finite group, and let $g \in G$. Define the *center* Z of G, and the *centralizer* $Z(g)$ of g. Prove that if g does not commute with every element of G, then

$$Z \;<\; Z(g) \;<\; G$$

where each inclusion is strict.

[5] Let G be a finite group of order p^n where p is prime. Define the *center* Z of G, and prove that Z has order greater than 1.

[6] Let p be prime. Prove that every group of order p^2 is abelian.

[7] For a finite group G and a prime p, let X be the set of p-tuples from G with product the identity:

$$X \;=\; \{\, (g_1, g_2, \ldots, g_p) \mid g_1 g_2 \cdots g_p = 1 \,\}$$

Let the cyclic group C_p act on X by rotation. In other words, let a generator of C_p take $(g_1, g_2, \ldots, g_p)$ to $(g_2, \ldots, g_p, g_1)$.

What is the order $|X|$ of X?

What are the possible orders of orbits $O_x \subset X$?

If p divides $|G|$, show that there is more than one orbit of order 1. Conclude that there exists an element $g \in G$ of order p.

[8] Draw a Cayley graph for the dihedral group D_4. Use this graph to illustrate the quotient $D_4/C_4 \cong C_2$.

[9] Find generators and relations for A_4, the alternating group of even permutations on four elements. For your choice of generators and relations, draw a Cayley graph for A_4.

[10] Prove *Burnside's formula*: If G is a finite group acting on a set X, then the number of orbits $|X/G|$ is equal to the average number of fixed points of the action,

$$|X/G| \;=\; \frac{1}{|G|}\sum_{g\in G} |X^g|$$

where $|X^g|$ is the number of elements of X fixed by g.

[11] How many different necklaces can be made from 9 red or blue beads, if we consider rotations to be the same necklace?

[12] There are 81 ways of coloring the four sides of a square, using three colors. How many distinct colorings are there, up to symmetry?

[13] There are $\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = 84$ ways of placing three checkers on a 3×3 checkerboard. How many distinct patterns are there, up to symmetry? Take into account the checkerboard pattern of light and dark squares.

[14] Using two colors, how many ways are there of coloring the elements of the Klein four-group $C_2 \times C_2$, up to symmetry?

(Hint: What is the automorphism group of $C_2 \times C_2$?)

[15] Let A_5 be the alternating group of even permutations on five elements. Define a *simple* group. Prove that A_5 is simple.

(You may assume that A_5 is isomorphic to the icosahedral group I of rotational symmetries of a dodecahedron, if you prefer working with that group.)

[Bonus] How many ways are there of painting the faces of a dodecahedron with two colors, up to rotation?

[Bonus] Let A_6 be the alternating group of even permutations on six elements. Prove that A_6 is simple.

(Hint: Don't bother analyzing how conjugacy classes of S_6 actually restrict to A_6. Instead, assume the worst case: Assume that if a conjugacy class of S_6 can split in half, then it does. Following a suggestion made in class, it is easier to separately consider the divisors of $|A_6|$ that are zero, or nonzero, mod 5.)

[Bonus] Let SL(2,7) be the group of 2 by 2 matrices with integer entries mod 7, and determinant 1.

What is the order of SL(2,7)?

What is the center Z of SL(2,7)?

Does SL(2,7) have any nontrivial normal subgroups, other than Z?

(This is not even a remotely fair question. If you think about it, it will completely derail you from studying for the exam!)