Practice Exam 1

Modern Algebra I, Dave Bayer, February 11, 2010

Our exam will consist of five questions, some asking for proofs and some asking for worked examples. What follows is the "universe" of problems from which I will draw these five questions. Some possibilities below are harder than I would ask on a test. You may nevertheless focus on what follows, in preparation for our exam. (The phrases in parentheses are useful search terms.)

[1] (List of small groups) List the distinct groups (up to isomorphism) of order < 12.

Each of the following problems are model questions with many possible variants, each based on groups of small order. On the actual exam, specific choices for G and H of order < 12 will be given.

[2] Let H be a subgroup of the group G. What are the right cosets of H in G?

[3] Let H be a subgroup of the group G. Is H a normal subgroup of G? What are the conjugate subgroups gHg^{-1} of H in G?

[4] Let G be a group. List the subgroups of G, and draw the lattice of subgroups by inclusion. Which subgroups are normal? What are the corresponding quotient groups?

[5] Let G and H be groups. How many group homomorphisms can you find from G to H? For each homomorphism, what is the kernel subgroup in G? What is the image subgroup in H?

[6] Let G be a group. What is the group of automorphisms (isomorphisms from G to G) of G?

Each of the following problems asks for a proof, for an arbitrary finite group G. If your proof depends on other results, you may state them without proof.

[7] (Coset) Let H be a subgroup of the group G, and let Ha, Hb be two right cosets of H in G. Show that either $Ha = Hb$ as sets, or they are disjoint.

[8] Let H be a subgroup of the group G. Show that the order of H divides the order of G.

[9] Let g be an element of the group G. Show that the order of g divides the order of G.

[10] (Fermat's little theorem) Let a be an integer, and p be a prime integer. Then $a^p \cong a \bmod p$.

[11] (Normal subgroup) Let $f : G \to H$ be a homomorphism of groups. Show that the kernel N of f is a normal subgroup of G.

[12] (Quotient group) Let N be a normal subgroup of the group G, and let G/N denote the set of right cosets Na of N in G. Show that the group operations from G are well-defined on G/N, so G/N forms a group.

[13] Let N be a normal subgroup of the group G. Show that N is the kernel of a map f for some group H and some group homomorphism $f : G \to H$.