*	*	*

Exam 1

Modern Algebra I, Dave Bayer, February 17, 2009

Name: Answers

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

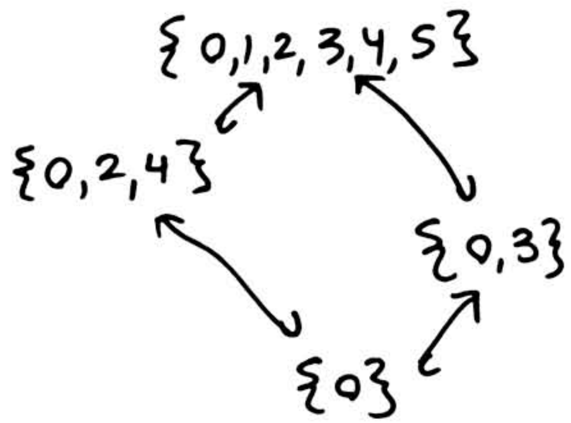
[1] Let C_6 be the cyclic group of order 6. Using generators and relations, we can write

$$C_6 \quad = \quad < a \mid a^6 = 1 >$$

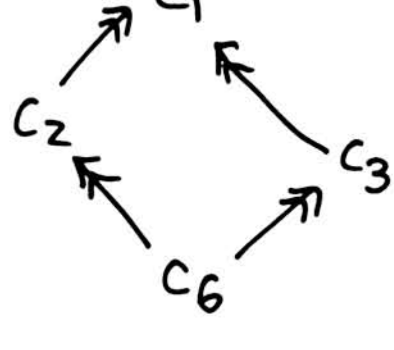
Working instead additively, we can write

$$(\ \mathbb{Z}/6\mathbb{Z}, +\ ) \quad = \quad \{\ 0,\ 1,\ 2,\ 3,\ 4,\ 5\ \} \quad \text{mod } 6$$

List the subgroups of C_6, and draw the lattice of subgroups by inclusion. Which subgroups are normal? What are the corresponding quotient groups? (You may use different notation if you prefer.)



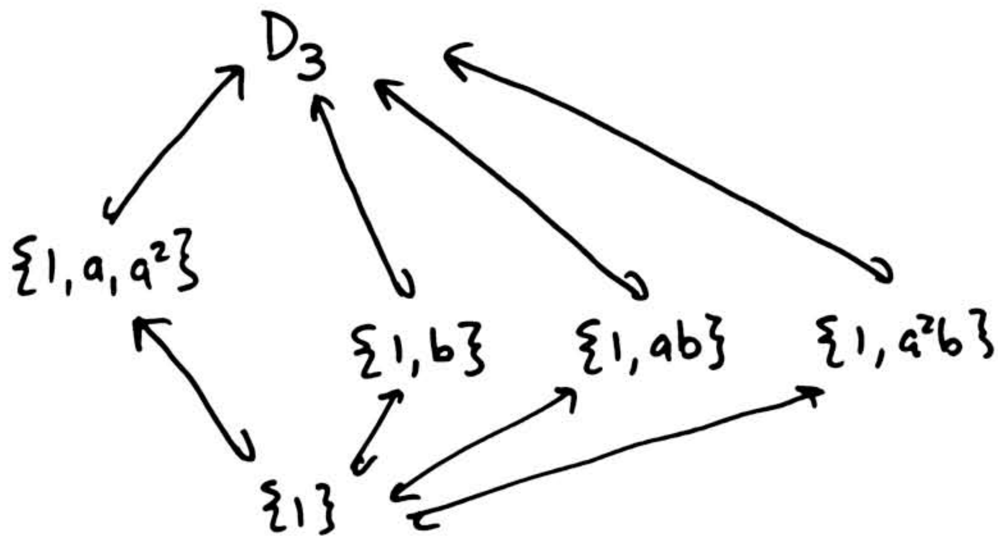
all normal, all quotients are cyclic:



[2] Let D_3 be the dihedral group of order 6 (the symmetries of the triangle). Using generators and relations, we can write

$$D_3 \quad = \quad < a, b \mid a^3 = 1,\ b^2 = 1,\ ba = a^{-1}b >$$

where a is a rotation and b is a flip. List the subgroups of G, and draw the lattice of subgroups by inclusion. Which subgroups are normal? What are the corresponding quotient groups? (You may use different notation if you prefer.)



identity, entire group and $\{1, a, a^2\}$ (rotations) are normal

$$D_3/\{1\} \cong D_3$$

$$D_3/D_3 \cong \{1\}$$

$$D_3/\{1, a, a^2\} \cong C_2 \quad \text{cyclic order 2}$$

[3] Let S_3 be the symmetric group of all permutations of {1,2,3}, given in cycle notation as

$$S_3 = \{ (), (12), (13), (23), (123), (132) \}$$

Let H be the subgroup of S_3 given by

$$H = \{ (), (123), (132) \}$$

What are the right cosets of H in S_3? What are the left cosets of H in S_3? Is H normal in S_3? If so, what is the quotient group S_3/H? If not, what are the conjugate subgroups gHg^{-1} of H in S_3? (You may use different notation if you prefer.)

right cosets

()	(12)
(123)	(23)
(132)	(13)

left cosets

()	(12)
(123)	(13)
(132)	(23)

} different order, same set

right and left cosets are the same, so H is normal.
(as it must be, index 2, so cosets are "H" and "not H")

Quotient group $S_3/H \cong C_2$ cyclic order 2

[4] Let A_4 be the alternating group of even permutations of {1,2,3,4}, given in cycle notation as

$$A_4 = \{ (), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243) \}$$

Let H be the subgroup of A_4 given by

$$H = \{ (), (123), (132) \}$$

What are the right cosets of H in A_4? What are the left cosets of H in A_4? Is H normal in A_4? If so, what is the quotient group A_4/H? If not, what are the conjugate subgroups gHg^{-1} of H in A_4? (You may use different notation if you prefer.)

right cosets

()	(12)(34)	(13)(24)	(14)(23)
(123)	(243)	(142)	(134)
(132)	(143)	(234)	(124)

left cosets

()	(12)(34)	(13)(24)	(14)(23)
(123)	(134)	(243)	(142)
(132)	(234)	(124)	(143)

not the same, so H is <u>not</u> normal.

gHg^{-1} is the same for any two g in same coset of H, so only need to try for g:

() H () = H (fixes 4)

(12)(34) {(), (123), (132)} (12)(34) = {(), (142), (124)} (must be inverse)

(H = {(), (123), (132)})

(relabels (1 2 3) → (2 1 4) = (142)) (fixes 3)

(13)(24) H (13)(24) = {(), (134), (143)} (fixes 2)

(341) (relabeled)

(14)(23) H (14)(23) = {(), (243), (234)} (fixes 1)

(432)

so conjugates are H, {(), (142), (124)}, {(), (134), (143)}, {(), (243), (234)}

[5] Let C_6 be the cyclic group of order 6, and let C_9 be the cyclic group of order 9. Using generators and relations, we can write

$$C_6 = < a \mid a^6 = 1 >$$
$$C_9 = < b \mid b^9 = 1 >$$

Working instead additively, we can write

$$(\mathbb{Z}/6\mathbb{Z}, +) = \{0, 1, 2, 3, 4, 5\} \mod 6$$
$$(\mathbb{Z}/9\mathbb{Z}, +) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \mod 9$$

How many group homomorphisms can you find from C_6 to C_9? How many group homomorphisms can you find from C_9 to C_6? (You may use different notation if you prefer.)

$C_6 \to C_9$ a must map to elem of order dividing 6

choices are $1, b^3, b^6$

$\{a \mapsto 1,\ a \mapsto b^3,\ a \mapsto b^6\}$

$C_9 \to C_6$ b must map to elem of order dividing 9

choices are $1, a^2, a^4$

$\{b \mapsto 1,\ b \mapsto a^2,\ b \mapsto a^4\}$

[6] Let $C_2 \times C_2$ be the Klein-4 group of order 4 (a product of two cyclic groups of order 2), and let D_4 be the dihedral group of order 8 (the symmetries of the square). Using generators and relations, we can write

$$C_2 \times C_2 = < a, b \mid a^2 = 1, b^2 = 1, ba = ab >$$
$$D_4 = < c, d \mid c^4 = 1, d^2 = 1, dc = c^{-1}d >$$

where c is a rotation and d is a flip. How many group homomorphisms can you find from $C_2 \times C_2$ to D_4? How many group homomorphisms can you find from D_4 to $C_2 \times C_2$? (You may use different notation if you prefer.)

$C_2 \times C_2 \to D_4$: a, b must map to elems of order dividing 2, that <u>commute</u> (could be the same elem)

Elems of $D_4 = \{1, c, c^2, c^3, d, cd, c^2d, c^3d\}$

order 1: 1; order 2: c^2, d, cd, c^2d, c^3d

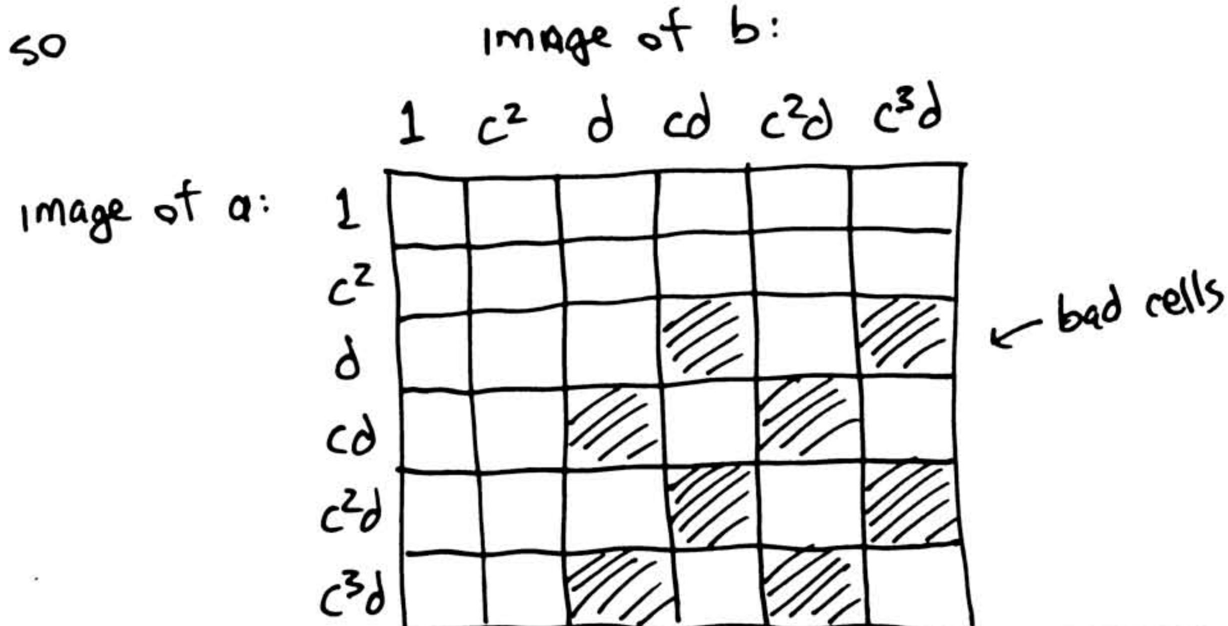
$1, c^2$ commute with everyone.

d, c^2d commute with each other: $c^2d \cdot d = c^2$, $dc^2d = c^2$

cd, c^3d " " " ": $cdc^3d = c^2$, $c^3dcd = c^2$

d, c^2d <u>don't</u> commute with cd, c^3d, e.g. $dcd = c^3$, $cdd = c$

so



There are $6 \cdot 6 - 8 = 28$ maps $C_2 \times C_2 \to D_4$ that are group homomorphisms.

(continued on back)

... [6] (continued from front)

$D_4 \to C_2 \times C_2$: Need images of c, d to also satisfy
$$c^4 = 1,\ d^2 = 1,\ dc = c^{-1}d$$

However, <u>every</u> elem of $C_2 \times C_2$ is order dividing 2, its own inverse, and commutes with all of $C_2 \times C_2$ (abelian).
So we can map c, d any way we like.

(4 choices for c) $\times$ (4 choices for d) =

16 homomorphisms $D_4 \to C_2 \times C_2$

---

check (since this seems a bit strange):
What are the quotients of D_4, and for each quotient, how many ways can we embed that quotient in $C_2 \times C_2$?

$D_4 / \{1\} \cong D_4$ (too big)

$D_4 / \{1, c^2\} \cong C_2 \times C_2$ (6 ways)

6 homomorphisms embedding into $C_2 \times C_2$, permuting 3 nonidentity elems any way we like.

$D_4 / \{1, c, c^2, c^3\} \cong C_2$ (3 ways)

$D_4 / \{1, c^2, d, c^2 d\} \cong C_2$ (3 ways)

$D_4 / \{1, c^2, cd, c^3 d\} \cong C_2$ (3 ways)

3 ways each to embed, choosing any nonidentity elem of $C_2 \times C_2$ for image of generator of C_2

$D_4 / D_4 \cong \{1\}$ (1 way) $1 \mapsto 1$

$6 + 3 + 3 + 3 + 1 = 16$ ✓