*	*	*

Practice Exam 2

Modern Algebra I, Dave Bayer, April 1, 2008

Name: Answers

[1] (6 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] How many different necklaces can be made from 6 red or blue beads, if we consider rotations to be the same necklace?

Burnside's Formula (p161) states $r \cdot |G| = \sum_{g \in G} |X_g|$

where r is # orbits, X_g are elements of X fixed by each g.

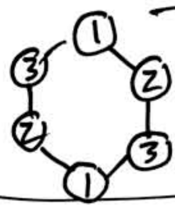
Our $G = C_6 = \{0,1,2,3,4,5\}$ +, mod 6

g	$\lvert X_g \rvert$
0	64
1	2
2	4
3	8
4	4
5	2

work: $|X_0| = |X|$, every pattern is fixed by identity.

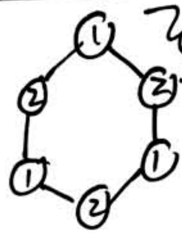
$|X| = 2^6$, either bead can be red or blue.

$|X_3| = 2^3$



3 choices, others forced by half-turn

$|X_2| = |X_4| = 2^2$



2 choices rest forced by 1/3 turn

$|X_1| = |X_5| = 2$, beads must be all red, or all blue

$$r = \frac{1}{6}(64+2+4+8+4+2) = \frac{1}{6}84 = \boxed{14 \text{ necklaces}}$$

[2] How many ways can 3 checkers be placed on a 4 by 4 checkerboard, if two arrangements are considered the same if they differ by a symmetry of the dihedral group D_4?

Again Burnside's Formula $r = \frac{1}{|G|} \sum_{g \in G} |X_g|$

g	$\lvert X_g \rvert$
1	560
$\circlearrowright$	0
$\curvearrowright$	0
$\circlearrowleft$	0
$\leftrightarrow$	0
$\updownarrow$	0
$\nwarrow\!\!\searrow$	28
$\swarrow\!\!\nearrow$	28

$\left.\begin{matrix}28\\28\end{matrix}\right\} 56$

$11 \cdot 56$

$|X_1| = \binom{16}{3} = \frac{16 \cdot 15 \cdot 14}{6} = 8 \cdot 5 \cdot 14 = 8 \cdot 70 = 560$

all possible patterns

$|X_{\circlearrowright}| = |X_{\circlearrowleft}| = 0$

1	2	3	1
3	4	4	2
2	4	4	3
1	3	2	1

checkers must be placed 4 at a time, we have 3

$|X_{\curvearrowright}| = 0$

1	2	3	4
5	6	7	8
8	7	6	5
4	3	2	1

2 at time

$|X_{\leftrightarrow}| = |X_{\updownarrow}| = 0$, same (2 at time)

$|X_{\nwarrow\searrow}| = |X_{\swarrow\nearrow}| = 28$

1	2	3	
4	5		3
6		5	2
	6	4	1

pick 2 at time off diag (6 choices)

and 1 at time on diagonal

$\left.\begin{matrix}1+1+1 & 4 \text{ ways} \\ 1+2 & 4 \cdot 6 \text{ ways}\end{matrix}\right\} 28 \text{ ways}$

$r = \frac{1}{8}(11 \cdot 56) = 11 \cdot 7 = \boxed{77 \text{ ways}}$

[3] The *Klein four* group V is the group of order 4 with elements

$$\{1, a, b, c\}$$

and the multiplication rules

$$a * a = b * b = c * c = 1, \quad a * b = b * a = c, \quad b * c = c * b = a, \quad c * a = a * c = b$$

1. Find two groups of order 10 which have the cyclic group C_5 of order 5 as a normal subgroup.
2. Find two groups of order 12 which have the Klein-4 group V of order 4 as a normal subgroup.

1. $C_5 \times C_2$, C_{10}, D_5 each have C_5 as normal subgroup

2. $V \times C_3$ (direct product) is abelian, w/ V as normal subgroup

For another let $\phi : V \to V$ be the "rotation"

$$\phi(1) = 1, \ \phi(a) = b, \ \phi(b) = c, \ \phi(c) = a$$

and take the semi-direct product

$$V \rtimes C_3 = \langle v, d \mid v \in V, d^3 = 1, \ dv = \phi(v) d \rangle$$

For a concrete example, $V = \{ (), (12)(34), (13)(24), (14)(23) \}$ is a normal subgroup of the alternating group A_4 of even permutations (order 12).

[4] The *Quaternion* group Q is the group of order 8 with elements

$$\{1, -1, i, -i, j, -j, k, -k\}$$

and the multiplication rules

$$i * i = j * j = k * k = -1, \quad i * j = -j * i = k, \quad j * k = -k * j = i, \quad k * i = -i * k = j$$

Find a nontrivial normal subgroup N of Q. For your choice of N, what is the quotient group Q/N?

$N = \{1, -1\}$ is normal in Q, with quotient $Q/N \cong V$ (Klein-4 group)

or

$H = \{1, i, -1, -i\}$ is normal in Q, with quotient $Q/H \cong C_2$

[5] Let Q be the Quaternion group of order 8, and let H be the cyclic subgroup of order 4 generated by the element i.

1. Is H a normal subgroup of Q? Why or why not?
2. Let X be the set of all 4-element subsets of Q. Let Q act on X by conjugation. H is an element of X; what is the size of its orbit?
3. How many orbits are there, for this action of Q on X?

1. Yes. Index 2, so only (right) cosets are H, and "not H". Same for left cosets, so right & left cosets agree $\Rightarrow$ H is normal.

2. The orbit of H is all conjugates of H.
Since H is normal, there is only one conjugate, H itself.
$\boxed{\text{size} = 1}$

3. ~~You're kidding, right?!~~ $r = \frac{1}{|G|} \sum_{g \in G} |X_g|$

$g \in Q$	$\lvert X_g \rvert$
1	70
-1	70
$\pm i$,	14
$\pm j$,	14
$\pm k$	14
	14
	14
	14

14·16

3 cases to consider.

$|X| = \binom{8}{4} = \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1} = 2\cdot 7\cdot 5 = 70$

$g = 1, -1$ commutes with every element $h \in Q$:

$gh = hg \Rightarrow ghg^{-1} = h$

$\Rightarrow gUg^{-1} = U$ for any $U \in X$

i is representative of other elements.

$$\left.\begin{aligned} i\,1(-i) &= 1 \\ i(-1)(-i) &= -1 \\ i(i)(-i) &= i \\ i(-i)(-i) &= -i \end{aligned}\right\} \text{fixed by conjugation}$$

$$\left.\begin{aligned} ij(-i) &= -j\,i(-i) = -j \\ &= \end{aligned}\right\} \text{sign flipped by conjugation.}$$

[5] continued.

..So i acts by conjugation on Q as follows:

$$\underbrace{1 \circlearrowleft \quad -1 \circlearrowleft \quad i \circlearrowleft \quad -1 \circlearrowleft}_{\text{choose one at time}} \qquad \underbrace{j \leftrightarrow -j \quad k \leftrightarrow -k}_{\text{choose in pairs}}$$

(This is just like checkerboard problem)

$4 = 1+1+1+1$	1 way (use all of $\pm 1, \pm i$)
$4 = 2+1+1$	$2\binom{4}{2}$ ways $= 12$ ways
$4 = 2+2$	1 way (use all of $\pm j, \pm k$)
	14 ways

$$r = \frac{1}{8} 14 \cdot 16 = 14 \cdot 2 = \boxed{28 \text{ orbits}}$$