Exam 3 (Take Home)

Modern Algebra I, Dave Bayer, April 26, 2008

Name: ____________________

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	[7] (5 pts)	[8] (5 pts)	TOTAL

This take home exam is out of 40 points, and our final exam will also be out of 40 points. They will bear a striking resemblence to each other. Answer as many or as few questions as you like on this take home exam. If you answer all eight questions, then this exam will be averaged with your final exam. If you answer fewer questions, then the contribution from this exam will be scaled. For example, if you answer two questions, then this take home exam will be worth 5 points, and the final exam will be worth 35 points.

I recommend that you answer all eight questions.

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[**1**] Let $f : G \to H$ be a homomorphism of groups with kernel N. Show that N is a normal subgroup of G.

[2] Let G be a finite group, and let the prime number p divide the order of G. Show that there exists an element $g \in G$ of order p.

[3] Let $f : G \to H$ be a homomorphism of groups with kernel N. Show that f maps two elements $g, h \in G$ to the same element of H if and only if g and h are in the same coset of N.

[**4**] Let H be a subgroup of G, and let N be a normal subgroup of G. Show that

$$\frac{HN}{N} = \frac{H}{H \cap N}$$

[5] Let G be a finite group of order p^n for a prime number p, and let G act on the finite set X. Show that

$$|X| \equiv |X_G| \mod p$$

[**6**] Let G be a finite group of order $p^n m$ for a prime number p, where p and m are relatively prime. Show that there exists a subgroup P of G of order p^n.

[7] Let G be a finite group of order $p^n m$ for a prime number p, where p and m are relatively prime. Let P and Q be two subgroups of G of order p^n. Show that P and Q are conjugate.

[**8**] Let G be a finite group of order $p^n m$ for a prime number p, where p and m are relatively prime. Let k be the number of subgroups of G of order p^n. Show that

$$k \equiv 1 \mod p$$