*	*	*

Exam 2

Modern Algebra I, Dave Bayer, April 1, 2008

Name: Solutions

[1] (6 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] How many different necklaces can be made from 12 red or blue beads, if we consider rotations to be the same necklace?

g	$\lvert X_g \rvert$	
0	2^{12}	4096
6	2^6	64
4,8	2^4	16,16
2,10	2^2	4,4
3,9	2^3	8,8
1,5,7,11	2	2,2,2,2

16,16 + 4,4: 40

8,8 + 2,2,2,2: 24

64 + 40 + 24: 128

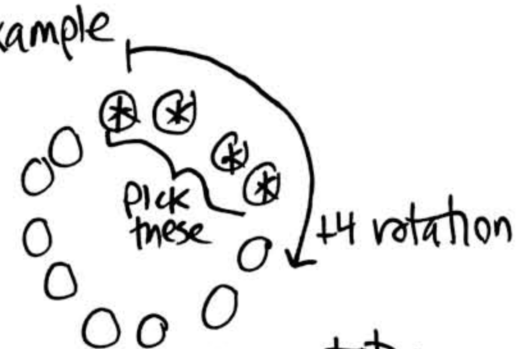
$$\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G} |X_g|, \quad G = C_{12}$$

$$\frac{1}{12}\,4224 = \frac{2112}{6} = \frac{704}{2} = 352$$

$$\boxed{352 \text{ necklaces}}$$

$$\begin{array}{r} 4096 \\ 128 \\ \hline 4224 \end{array} \checkmark \qquad \begin{array}{r} 352 \\ 12 \\ \hline 704 \\ 352 \\ \hline 4224 \end{array} \checkmark$$

example



rest decided by rotating

[2] How many ways can 4 checkers be placed on a 4 by 4 checkerboard, if two arrangements are considered the same if they differ by a symmetry of the dihedral group D_4?

g	$\lvert X_g \rvert$	
1	1820	1848
↷	28	
↷, ↶	4, 4	$3 \cdot 56 = 168$
⊕, ⊕	28, 28	
⊠, ⊠	52, 52	

$$\begin{array}{r} 1848 \\ 168 \\ \hline 2016 = \end{array}$$

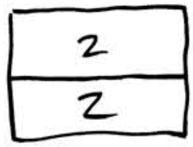
$$\frac{1}{8} 2016 = 252$$

$$\boxed{252 \text{ arrangements}}$$

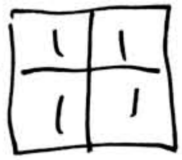
$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} |X_g|, \quad G = D_4$$

$$X = \binom{16}{4} = \frac{16 \cdot 15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \underbrace{\underbrace{4 \cdot 5}_{20} \cdot \underbrace{7 \cdot 13}_{91}}_{1820}$$

91, my favorite prime!

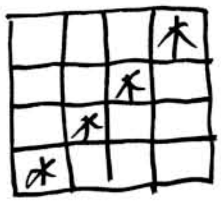
↷ 

top replicates in bottom,
pick ~~4~~ from top 8,
forces other 2 from bottom

↷ 

$$\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 4 \cdot 7 = 28$$

$$\binom{4}{1} = 4$$

⊠, ⊠ 

1 at time on diag 4
2 at time off diag 6

$$\begin{aligned} 4 &= 1+1+1+1 & & 1 \\ &= 2+1+1 & & 6\binom{4}{2} = 36 \\ &= 2+2 & & \binom{6}{2} = \frac{6 \cdot 5}{2 \cdot 1} = 15 \end{aligned} \Bigg\} 52$$

1. Find two groups of order 8 which have the cyclic group C_4 of order 4 as a normal subgroup.
2. Find two groups of order 18 which have the symmetric group S_3 of order 6 as a normal subgroup.

1. $C_4 \times C_2$, $D_4 = \langle a, b \mid a^4 = 1, b^2 = 1, ba = a^{-1}b \rangle$

2. $S_3 \times C_3$

For second, want $S_3 \leq G$ normal subgroup so
$$G/S_3 \cong C_3$$

semi-direct product $G = \langle a, b \mid a \in S_3, b^3 = 1, ba = \phi(a) b \rangle$
where $\phi : S_3 \to S_3$ is an automorphism of S_3 of order 3.

One such automorphism is to conjugate by (123):

$$g \overset{\phi}{\longmapsto} (123)\, g\, (132)$$

$$() \longmapsto ()$$

$$(12) \longmapsto (123)(12)(132) = (13)$$

1 — 2 —— 1 — 3
2 — 3 —— 2
3 — 1 — 2 —— 1

or apply (132) to letters of (12):

$$\begin{array}{c} (1\ 2) \\ {\scriptstyle (132)}\ \downarrow\ \downarrow \\ (3\ 1) = (13) \end{array}$$

$$(13) \longmapsto (23)$$
$$(23) \longmapsto (12)$$
$$(123) \longmapsto (123)$$
$$(132) \longmapsto (132)$$

So ϕ fixes $(), (123), (132)$ and "rotates"

$$(12) \longmapsto (13) \longmapsto (23) \longmapsto (12)$$

continued

[4] Let A_4 be the alternating group of order 12, of all even permutations of $\{1, 2, 3, 4\}$. Find a nontrivial normal subgroup N of A_4. For your choice of N, what is the quotient group A_4/N?

Let $N = \{ (), (12)(34), (13)(24), (14)(23) \}$

N consists of two complete conjugacy classes in S_4, where the shape of a permutation alone determines conjugacy. With fewer elements in A_n than S_n, conjugacy classes can only get smaller (here, they actually stay the same) so N consists of complete conjugacy classes in A_4, and is normal.

$A_4/N \cong C_3$ since it is order 3, C_3 is only possibility.

[5] Let A_4 be the alternating group of order 12, of all even permutations of $\{1,2,3,4\}$. Let H be the cyclic subgroup of order 3 generated by the permutation $(1\,2\,3)$.

1. Is H a normal subgroup of A_4? Why or why not?
2. Let X be the set of all 3-element subsets of A_4. Let A_4 act on X by conjugation. H is an element of X; what is the size of its orbit?
3. How many orbits are there, for this action of A_4 on X?

1) $H = \{(), (123), (132)\}$

$(12)(34)\,H\,(12)(34) = \{(), (12)(34)\underbrace{(123)}_{\text{square}}(12)(34), \text{''}\}$

$(12)(34)(123)(12)(34) = (142)$ $\quad \not\Vert \quad H$

$\downarrow\downarrow\downarrow$ 2 1 4

so $H \neq gHg^{-1}$ for $g=(12)(34)$

$\boxed{\text{H is not normal}}$

---

2) $A_4 = \{\underbrace{()}_{1}, \underbrace{(12)(34), \ldots}_{3}, \underbrace{(123)}_{8}\}$

and the 8 3-cycles come in 4 pairs, related by squaring.
So H has at <u>most</u> 4 conjugates, counting H itself.

It would appear that for any $g \notin H$, $gHg^{-1} \neq H$ but
$g \in H$, $gHg^{-1} = H$ (H commutes with itself)

so conjugates of H correspond to cosets of $\underset{3}{H}$ in $\underset{12}{A_4}$: 4.

check explicitly we can reach each one:

$(1\,2\,3) \xrightarrow{(12)(34)} (2\,1\,4) = (142)$

$(1\,2\,3) \xrightarrow{(13)(24)} (3\,4\,1) = (134)$

$(1\,2\,3) \xrightarrow{(14)(23)} 4\,3\,2 = (243)$

yes ✓

$\boxed{\text{Orbit has size 4}}$

(5, continued) (p6)

3) ~~You're kidding, right?!~~

$$|X| = \binom{12}{3} = \frac{12\cdot 11\cdot 10}{6} = 220 \text{ subsets.}$$

A_4 consists of 3 element types: $()$, (123), $(12)(34)$
$\qquad\qquad\qquad\qquad\qquad\qquad 1 \qquad 8 \qquad 3$

$$\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G}|X_g| \quad \text{where } |X_{()}| = 220$$

We simply need to count $|X_{(123)}|$ and $|X_{(12)(34)}|$

For each of (123) and $(12)(34)$, how do they act on A_4 by conjugation?

---

(123):

$$() \circlearrowleft \quad (123) \circlearrowleft \quad (132) \circlearrowleft$$

$$\begin{tikzcd}[column sep=small] & (124) \arrow[dr] & \\ (\cdots) \arrow[ur] & & (\cdots) \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (\cdots) \arrow[dr] & \\ (\cdots) \arrow[ur] & & (\cdots) \arrow[ll] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & (12)(34) \arrow[dr] & \\ (\cdot\cdot)(\cdot\cdot) \arrow[ur] & & (\cdot\cdot)(\cdot\cdot) \arrow[ll] \end{tikzcd}$$

$(12)(34)$:

$$() \circlearrowleft \quad (12)(34) \circlearrowleft \quad (13)(24) \circlearrowleft \quad (14)(23) \circlearrowleft$$

$$(123) \leftrightarrow (\cdots)$$
$$(\cdots) \leftrightarrow (\cdots)$$
$$(\cdots) \leftrightarrow (\cdots)$$
$$(\cdots) \leftrightarrow (\cdots)$$

---

(123): $3 = 1+1+1$ | 1
$\qquad = 3$ | 3

$|X_{(123)}| = 4$

$(12)(34)$: $3 = 1+1+1$ | 4
$\qquad = 1+2$ | $4\cdot 4 = 16$

$|X_{(12)(34)}| = 20$

(5, continued) p7

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{12}\left(|X_{()}| + 8|X_{(123)}| + 3|X_{(12)(34)}|\right)$$

$$= \frac{1}{12}\left(220 + \underbrace{8\cdot 4}_{32} + \underbrace{3\cdot 20}_{60}\right)$$

$$= \frac{1}{12}(312) = 26$$

3) $\boxed{\text{26 orbits}}$