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Exam 1

Modern Algebra I, Dave Bayer, February 19, 2008

Name: Answers

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] Complete each of the following multiplication tables, so that the resulting table is the multiplication rule for a group G. In each case, what group do you get?

*	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

$C_2 \times C_2$

$\mapsto$	
1	(0,0)
2	(1,0)
3	(0,1)
4	(1,1)
*	+

*	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

C_4

1	0
2	2
3	1
4	3
*	+

*	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	5	6	4
3	3	1	2	6	4	5
4	4	6	5	1	3	2
5	5	4	6	2	1	3
6	6	5	4	3	2	1

order 6, not abelian

$\Rightarrow S_3$

[2] For each of the groups found in problem 1, let the group act on itself by right multiplication. Use these actions to find permutation representations for each group.

$C_2 \times C_2$

	column of table	permutation in S_4
1	1 2 3 4	()
2	2 1 4 3	(12)(34)
3	3 4 1 2	(13)(24)
4	4 3 2 1	(14)(23)

C_4

		in S_4
1	1 2 3 4	()
2	2 1 4 3	(12)(34)
3	3 4 2 1	(1324)
4	4 3 1 2	(1423)

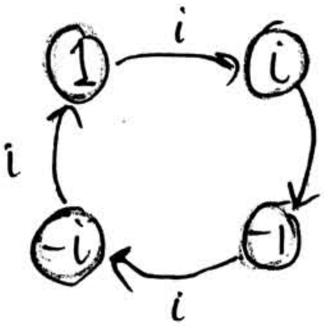
S_3

		in S_6
1	1 2 3 4 5 6	()
2	2 3 1 6 4 5	(123)(465)
3	3 1 2 5 6 4	(132)(456)
4	4 5 6 1 2 3	(14)(25)(36)
5	5 6 4 3 1 2	(15)(26)(34)
6	6 4 5 2 3 1	(16)(24)(35)

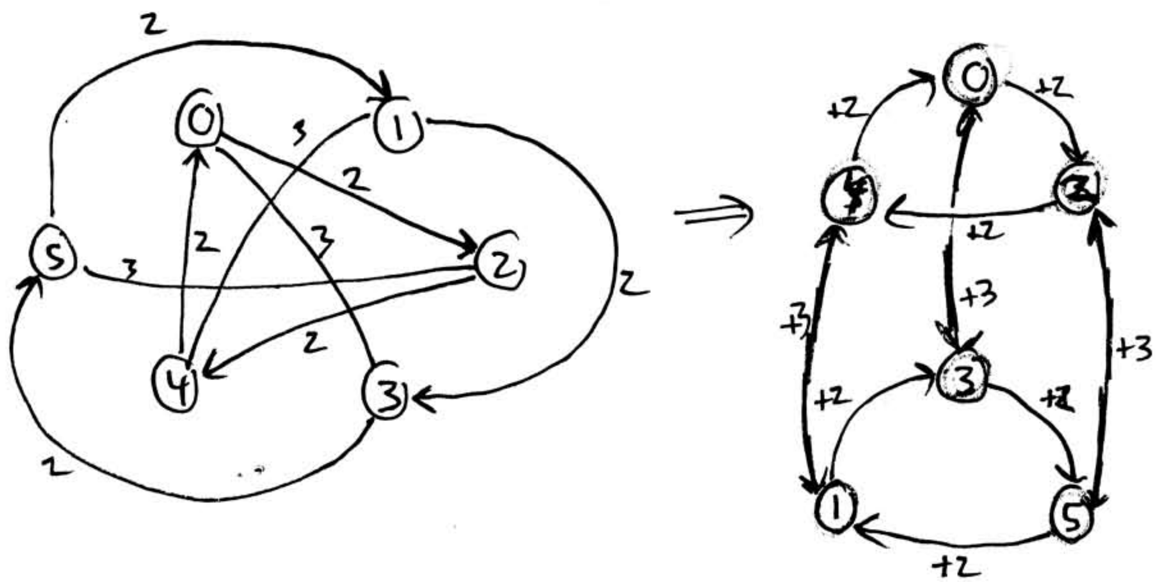
[3] Using the given generators, draw the Cayley graphs for each of the following groups.

1. $G = \langle \{1, i, -1, -i\}, * \rangle \subset \langle \mathbb{C}, * \rangle$, using the generator i.
2. $G = \langle \mathbb{Z}/6\mathbb{Z}, + \rangle$, using the generators 2 and 3.
3. $G = S_3$, using the generators $(1\ 2\ 3)$ and $(1\ 2)$.

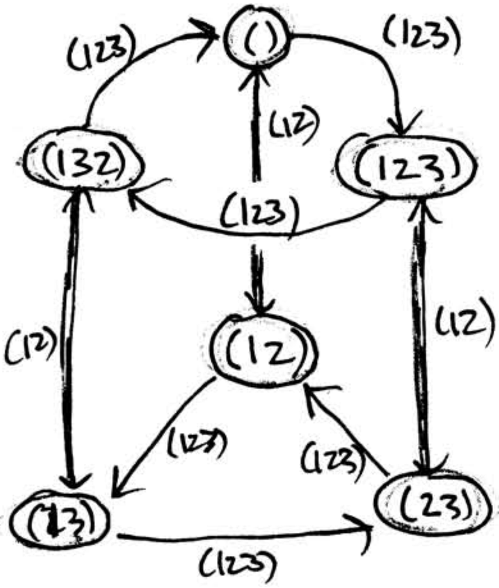
1)



2)



3)



$(123)(12) = (23)$

$(12)(123) = (13)$

[4] Here is a multiplication table for the dihedral group D_5 of order 10:

*	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	1	7	8	9	10	6
3	3	4	5	1	2	8	9	10	6	7
4	4	5	1	2	3	9	10	6	7	8
5	5	1	2	3	4	10	6	7	8	9
6	6	10	9	8	7	1	5	4	3	2
7	7	6	10	9	8	2	1	5	4	3
8	8	7	6	10	9	3	2	1	5	4
9	9	8	7	6	10	4	3	2	1	5
10	10	9	8	7	6	5	4	3	2	1

1. Find six nontrivial subgroups H of D_5 (not order 1 or 10).
2. For two of the H that you found, what are the right cosets of H in G?

right coset Hb
$\Leftrightarrow$ left action by H

1) $\{1,2,3,4,5\}$

$\{1,6\}$, $\{1,7\}$, $\{1,8\}$, $\{1,9\}$, $\{1,10\}$

2) $\{1,6\}b$ for b in group

$\{1,6\}1 = \{1,6\} = \{1,6\}6$

$\{1,6\}2 = \{2,10\} = \{1,6\}10$

$\{1,6\}3 = \{3,9\} = \{1,6\}9$

$\{1,6\}4 = \{4,8\} = \{1,6\}8$

$\{1,6\}5 = \{5,7\} = \{1,6\}7$

cosets of $\{1,6\}$

cosets of $\{1,2,3,4,5\}$

$\{1,2,3,4,5\}\,1 = \{1,2,3,4,5\}$

$\{1,2,3,4,5\}\,6 = \{6,7,8,9,10\}$

[5] Let G be the group $\langle \mathbb{Z}/3\mathbb{Z}, + \rangle$, and let H be the group $\langle \mathbb{Z}/6\mathbb{Z}, + \rangle$. How many group homomorphisms can you find from G to H? From H to G?

$G = \{0, 1, 2\} \bmod 3$

$H = \{0, 1, 2, 3, 4, 5\} \bmod 6$

$\boxed{G \to H \quad \text{3 maps}}$

$1+1+1=0$, need also for image

$1 \to 0$ $\quad$ $1 \mapsto 2$ $\quad$ $1 \mapsto 4$

generator 1 determines map

$\mapsto$	
0	0
1	0
2	0

$\mapsto$	
0	0
1	2
2	4

$\mapsto$	
0	0
1	4
2	2

$\boxed{H \to G \quad \text{3 maps}}$

$1+1+1+1+1+1 = 0$, need also for image

anything works!

generator 1 determines map

$\to$	
0	0
1	0
2	0
3	0
4	0
5	0

$\to$	
0	0
1	1
2	2
3	0
4	1
5	2

$\to$	
0	0
1	2
2	1
3	0
4	2
5	1

[6] The *Klein four* group V is the group of order 4 with elements

$$\{1, a, b, c\}$$

and the multiplication rules

$$a * a = b * b = c * c = 1, \quad a * b = b * a = c, \quad b * c = c * b = a, \quad c * a = a * c = b$$

The *Quaternion* group Q is the group of order 8 with elements

$$\{1, -1, i, -i, j, -j, k, -k\}$$

and the multiplication rules

$$i * i = j * j = k * k = -1, \quad i * j = -j * i = k, \quad j * k = -k * j = i, \quad k * i = -i * k = j$$

How many group homomorphisms can you find from V to Q? From Q to V?

$\boxed{\begin{array}{c} V \to Q \\ \text{4 maps} \end{array}}$

first, need $a*a = b*b = c*c = 1$

only possibilities ~~is~~ are $a, b, c \mapsto -1$ or 1

in all cases, image is $\{1\}$ or $\{1, -1\}$

i, j, k not in image

1	1	~~1~~	~~1~~	1	~~1~~	1	1	~~1~~
a	1	~~1~~	~~1~~	1	~~-1~~	-1	-1	~~-1~~
b	1	~~1~~	~~-1~~	-1	~~1~~	1	-1	~~-1~~
c	1	~~-1~~	~~1~~	-1	~~1~~	-1	1	~~-1~~

do these all work? need $a*b = c$, etc.

these fail

(continued on next page)

(problem 6, continued)

$\boxed{\begin{array}{c} Q \to V \\ 16 \text{ maps} \end{array}}$

$i * j = -j * k$, but V is abelian

$\Rightarrow -1, 1$ must be same

$-i, i$ $\quad -j, j$ $\quad -k, k$ each same

so it comes down to where does i, j, k map?

up to sign, any two product is third,

just like a, b, c

so $\underline{\underline{6}}$ permutations $i, j, k \mapsto a, b, c$

1 trivial map $i, j, k \mapsto 1$

what about subgroups $i, j \mapsto a$, $k \to 1$?

choose any of i, j, k to send to 1 (3x)

send rest to one of a, b, c (3x)

so

± 1	1	1	1
$\pm i$	1	a	1
$\pm j$	1	b	a
$\pm k$	1	c	a
	trivial	six versions	nine versions