Additional Practice Problems

Linear Algebra, Dave Bayer, May, 2004

Name: Answers

Please work only one problem per page, starting with the pages provided, and identify all continuations clearly.

[1] Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D.

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answer:

$$\underset{S \leftarrow S}{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}} = \underset{S \leftarrow E}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{E \leftarrow E}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{E \leftarrow S}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}}$$

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work:

$$\det|A - \lambda I| = \lambda^2 - \underbrace{\text{trace}(A)}_{4+1}\lambda + \underbrace{\det(A)}_{4\cdot 1-(-2)1} = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0 \quad \boxed{\lambda = 2, 3}$$

$$\lambda = 2: \quad A - 2I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boxed{\lambda_1 = 2,\ v_1 = (1,1)}$$

$$\lambda = 3: \quad A - 3I: \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boxed{\lambda_2 = 3,\ v_2 = (2,1)}$$

inverse ✓

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} -2 & 4 \\ 3 & -3 \end{bmatrix}} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} ✓$$

v_1 v_2

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[2] Let $A = \begin{bmatrix} -5 & 3 & 4 \\ -2 & 2 & 2 \\ -6 & 3 & 5 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D.

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answer:

$$\underset{S \leftarrow S}{\begin{bmatrix} -5 & 3 & 4 \\ -2 & 2 & 2 \\ -6 & 3 & 5 \end{bmatrix}} = \underset{S \leftarrow E}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}} \underset{E \leftarrow E}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{E \leftarrow S}{\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}}$$

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work:

$$-\det|A-\lambda I| = \lambda^3 - \underbrace{\text{trace}(A)}_{\substack{-5+2+5 \\ 2}}\lambda^2 + \underbrace{\text{trace}(\Lambda^2 A)}_{\substack{\begin{vmatrix} -5 & 3 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} -5 & 4 \\ -6 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} \\ -4 \qquad -1 \qquad 4}}\lambda - \det(A) = 0$$

$$\det(A) = -5\underset{4}{\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix}} + 2\underset{3}{\begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix}} - 6\underset{-2}{\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix}} = -20+6+12 = -2$$

$$\begin{matrix} + \\ - \\ + \end{matrix}$$

λ	-2	-1	0	1	2
$\lambda^3-2\lambda^2-\lambda+2$	-12	0	2	0	0

roots

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$\underbrace{(\lambda+1)(\lambda-1)}_{\lambda^2-1}(\lambda-2) = 0$$

$\cdot$	λ	-2
λ^2	λ^3	$-2\lambda^2$
-1	$-\lambda$	2

$$\boxed{\lambda = -1, 1, 2}$$

$$\lambda=-1: \begin{bmatrix} -4 & 3 & 4 \\ -2 & 3 & 2 \\ -6 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{0} \qquad \lambda=1: \begin{bmatrix} -6 & 3 & 4 \\ -2 & 1 & 2 \\ -6 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \lambda=2: \begin{bmatrix} -7 & 3 & 4 \\ -2 & 0 & 2 \\ -6 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & & \\ & 1 & \\ & & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}/1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -1 \\ -4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 4 \\ -2 & 2 & 2 \\ -6 & 3 & 5 \end{bmatrix}$$

inverse

1	0	1	1	0
1	2	0	1	2
1	1	1	1	1
1	0	1	1	0
1	2	0	1	2

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[**3**] Let $A = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$. Find the matrix exponential e^{At}.

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answer:

$$e^{At} = \begin{bmatrix} 2te^{2t}+e^{2t} & -4e^{2t} \\ te^{2t} & e^{2t}-2te^{2t} \end{bmatrix}$$

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work:

$$\det|A-\lambda I| = \lambda^2 - \underbrace{\text{trace}(A)}_{4+0}\lambda + \underbrace{\det(A)}_{4\cdot 0-(-4)\cdot 1} = \lambda^2-4\lambda+4 = (\lambda-2)^2 = 0$$

$$\boxed{\lambda = 2,2}$$

$\lambda = 2$: $B = A - 2I = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$ <u>Can't</u> find independent v_1, v_2 so $v_1, v_2 \xrightarrow{B} 0$

Instead try to find $v_2 \xrightarrow{B} v_1 \xrightarrow{B} 0$

$v_2 = (0,1)$? $Bv_2 = (-4,-2)$ too messy

$v_2 = (1,0)$ $Bv_2 = (2,1) = v_1$ $Bv_1 = \vec{0}$ ☑

$$\begin{array}{cc} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ \uparrow\ \ \uparrow \\ v_1\ \ v_2 \end{array} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix}} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}}_{\begin{bmatrix} te^{2t} & e^{2t}-2te^{2t} \\ e^{2t} & -2e^{2t} \end{bmatrix}} = \begin{bmatrix} 2te^{2t}+e^{2t} & -4te^{2t} \\ te^{2t} & e^{2t}-2te^{2t} \end{bmatrix}$$

☑

$$e^{At}\Big|_{t=0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{I}{☑}$$

$$\frac{d}{dt}e^{At}\Big|_{t=0} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \overset{A}{☑}$$

or

$$e^{2t}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}}_{\begin{bmatrix} t & 1+2t \\ 1 & -2 \end{bmatrix}} = e^{2t}\begin{bmatrix} 2t+1 & -4t \\ t & 1-2t \end{bmatrix}$$

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[4] Let $A = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 2 & 3 \\ -4 & 1 & 3 \end{bmatrix}$. Find the matrix exponential e^{At}.

answer:

$$e^{At} = \begin{bmatrix} 3-2e^t-te^t & -1+e^t & -1+e^t+te^t \\ 3-3e^t-2te^t & -1+2e^t & -1+e^t+2te^t \\ 3-3e^t-te^t & -1+e^t & -1+2e^t+te^t \end{bmatrix}$$

work:

$$-\det|A-\lambda I| = \lambda^3 - \underbrace{\text{trace}(A)}_{-3+2+3}\lambda^2 + \underbrace{\text{trace}(\Lambda^2 A)}_{\begin{vmatrix}-3&1\\-5&2\end{vmatrix} + \begin{vmatrix}-3&2\\-4&3\end{vmatrix} + \begin{vmatrix}2&3\\1&3\end{vmatrix}}\lambda - \det(A),$$

$$\qquad\qquad -1 \qquad -1 \qquad 3$$

$\det(A)$ ← 0, because A singular: $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{0}$ hmm...

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$
$$\lambda(\lambda-1)^2 = 0 \qquad \boxed{\lambda = 0, 1, 1}$$

$\lambda=0$: $\begin{bmatrix} -3 & 1 & 2 \\ -5 & 2 & 3 \\ -4 & 1 & 3 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{0}$ $\quad \boxed{v_1 = (1,1,1)}$

$\lambda=1$: $\begin{bmatrix} -4 & 1 & 2 \\ -5 & 1 & 3 \\ -4 & 1 & 2 \end{bmatrix} = B = A - 1I$

Rank 2, can't find independent v_2, v_3 so $v_2, v_3 \xrightarrow{B} 0$

Insteadd try to find $v_3 \xrightarrow{B} v_2 \xrightarrow{B} 0$

Most v_3 so $B^2 v_3 = 0$ will work.

$B^2 = \begin{bmatrix} 3 & -1 & -1 \\ 3 & -1 & -1 \\ 3 & -1 & -1 \end{bmatrix}$ Try $v_3 = (0,1,-1)$

$Bv_3 = v_2 = (-1,-2,-1) \quad Bv_2 = 0$

$\boxed{v_2 = (1,2,1), \; v_3 = (0,-1,1)}$ (negate to clean up)

Invert ✓

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ \uparrow\;\uparrow\;\uparrow \\ v_1\; v_2\; v_3 \end{array} \underbrace{\begin{bmatrix} 0 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -2 & +1 & 1 \\ -1 & 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 2 & 3 \\ -4 & 1 & 3 \end{bmatrix} \checkmark$$

$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 \end{array}$$

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[4] continued:

$$e^{\begin{bmatrix}0&&\\&1&1\\&&1\end{bmatrix}t} = \begin{bmatrix}1&&\\&e^t&te^t\\&&e^t\end{bmatrix}$$

$\boxed{\text{temporarily set } a=e^t,\ b=te^t}$

$$e^{At} = \begin{bmatrix}1&1&0\\1&2&-1\\1&1&1\end{bmatrix}\underbrace{\begin{bmatrix}1&&\\&a&b\\&&a\end{bmatrix}\begin{bmatrix}3&-1&-1\\-2&1&1\\-1&0&1\end{bmatrix}}_{\begin{bmatrix}3&-1&-1\\-2a+b&a&a+b\\-a&0&a\end{bmatrix}} = \begin{bmatrix}3-2a-b&-1+a&-1+a+b\\3-3a-2b&-1+2a&-1+a+2b\\3-3a-b&-1+a&-1+2a+b\end{bmatrix}$$

$t=0$: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ✓

$\frac{d}{dt}$ then $t=0$: $\begin{bmatrix}-3&1&2\\-5&2&3\\-4&1&3\end{bmatrix}$ ✓

Check: $e^{At}\big|_{t=0} = I$: $a|_{t=0} = e^t|_{t=0} = 1$, $b|_{t=0} = te^t|_{t=0} = 0$

$\left(\frac{d}{dt}e^{At}\right)_{t=0} = A$: $\frac{da}{dt}\big|_{t=0} = e^t|_{t=0} = 1$, $\frac{db}{dt}\big|_{t=0} = (e^t+te^t)\big|_{t=0} = 1$

(constants $\to 0$!)

so $$e^{At} = \begin{bmatrix}3-2e^t-te^t & -1+e^t & -1+e^t+te^t\\3-3e^t-2te^t & -1+2e^t & -1+e^t+2te^t\\3-3e^t-te^t & -1+e^t & -1+2e^t+te^t\end{bmatrix}$$

(copy <u>carefully</u> to answer zone)

check copying by comparing coefficients; top of this page against answer.

$$\begin{bmatrix}3,-2,-1 & -1,1,0 & -1,1,1\\3,-3,-2 & -1,2,0 & -1,1,2\\3,-3,-1 & -1,1,0 & -1,2,1\end{bmatrix}$$

(Another approach would be to <u>find</u> these coefficients by solving 9 3x3 systems of equations, probably by eye.)