# Exam 1

Linear Algebra, Dave Bayer, February 17, 2004

**Name:** ANSWER KEY

| [1] (5 pts) | [2] (5 pts) | [3] (5 pts) | [4] (5 pts) | [5] (5 pts) | [6] (5 pts) | TOTAL |
|---|---|---|---|---|---|---|
| | | | | | | |

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

**[1]** What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$(0,0,1,0)$ is a particular solution, by inspection, because $3^{rd}$ column = right hand side.

problem row reduces to (using 0 right hand side, not shown)

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

so 2 independent homog solutions are $(1,1,0,0)$, $(0,0,1,-1)$

answer =

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} t$$



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[**6**] What is the determinant of the following $8 \times 8$ matrix?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

$\boxed{\det = 39}$

| n | F(n) |
|---|---|
| 1 | 1 |
| 2 | -1 |
| 3 | 1 |
| 4 | 7 |
| 5 | 1 |
| 6 | -9 |
| 7 | 17 |
| 8 | 39 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | |
| 2 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | |
| 2 | 0 | -1 | -1 | 1 | 0 | 0 | 0 | 0 | 2 - 2 1 |
| 3 | 0 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | |
| 2 | 0 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 2 + 3 |
| 3 | 0 | 0 | 1 | -3 | -1 | 0 | 0 | 0 | 3 - 2 2 |
| 4 | 0 | 0 | 2 | 1 | 1 | 1 | 0 | 0 | |
| 4 | 0 | 0 | 0 | 7 | 3 | 1 | 0 | 0 | 4 - 2 3 |
| 5 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | 0 | |
| 4 | 0 | 0 | 0 | 1 | 0 | -2 | -3 | 0 | 4 - 3 5 |
| 5 | 0 | 0 | 0 | 0 | 1 | 5 | 7 | 0 | 5 - 2 4 |
| 6 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | 1 | |
| 6 | 0 | 0 | 0 | 0 | 0 | -9 | -13 | 1 | 6 - 2 5 |
| 7 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | -8 | 6 | 6 + 5 7 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 17 | -11 | 7 - 2 6 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -19 | 7 - 8 8 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 39 | 8 - 2 7 |



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**[6]** What is the determinant of the following $8 \times 8$ matrix?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

$F(n) = n \times n$ det

recurse on column expansion, $n=5$

$$F(5) = +1 \underbrace{\begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix}}_{F(4)} - 2 \underbrace{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix}}_{\left( +1 \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix}}_{F(3)} - 2 \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix}}_{F(2)} \right)}$$

$1(-1) - 2(-1) = 1$

| $n$ | $F(n)$ | |
|---|---|---|
| 2 | $-1$ | by direct calculation |
| 3 | 1 | |
| 4 | 7 | |
| 5 | 1 | $7 - 2 \cdot 1 + 4(-1)$ |
| 6 | $-9$ | $1 - 2 \cdot 7 + 4 \cdot 1$ |
| 7 | 17 | $-9 - 2 \cdot 1 + 4 \cdot 7$ |
| 8 | 39 | $17 - 2(-9) + 4 \cdot 1$ |

$$\boxed{F(n) = F(n-1) - 2F(n-2) + 4\,F(n-2)}$$

$$\boxed{\det = 39}$$



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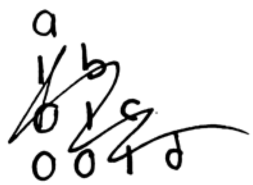
[5] Give a formula for the matrix which is inverse to:

$$I = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & d \end{bmatrix} \begin{bmatrix} bcd & -cd & d & -1 \\ 0 & acd & -ad & a \\ 0 & 0 & abd & -ab \\ 0 & 0 & 0 & abc \end{bmatrix} \Big/ abcd$$



eyeball it

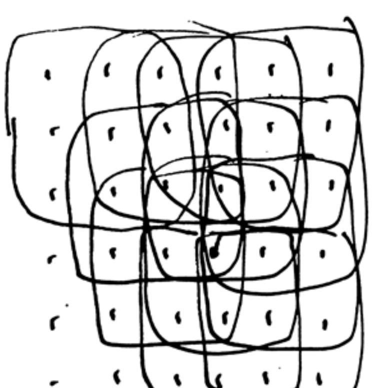
- guess upper triangular
- write down what diagonal entries must be
- force above diagonal, in stages

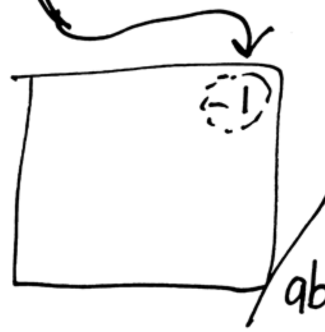
or

$$\begin{array}{c|cccccc} a & 0 & 0 & 0 & a & 0 & 0 \\ \hline 1 & b & 0 & 0 & 1 & b & 0 \\ 0 & 1 & c & 0 & 0 & 1 & c \\ 0 & 0 & 1 & d & 0 & 0 & 1 \\ a & 0 & 0 & 0 & a & 0 & 0 \\ 1 & b & 0 & 0 & 1 & b & 0 \\ 0 & 1 & c & 0 & 0 & 1 & c \end{array}$$

read out overlapping 3x3 minors, apply checkerboard of signs (since 4x4 is even case, don't need for odd cases, eg 3x3)

| | | | |
|---|---|---|---|
| + | − | + | − |
| | + | − | + |
| | | + | − |
| | | | + |





/abcd

(or row reduce!)



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[2] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xRightarrow{① \leftrightarrow ②} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{③ = ③ - ①} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xRightarrow{② \rightleftarrows ③} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xRightarrow{③ = \frac{1}{2} ③} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xRightarrow{② = ② - ③} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

answer



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[3] What is the determinant of the following $4 \times 4$ matrix?

expand down first col

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$+1 \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix}}_{\text{3x3 case, same pattern}} - 2 \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix}}_{}$$

$$\underbrace{\left( +1 \underbrace{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}_{\text{2x2 case}} - 2 \underbrace{\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}_{\text{triangular, det = 1}} \right)}_{-3}$$

$$\underbrace{\left( +1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \right)}_{1}$$

$$1 \cdot 1 - 2(-3) = \boxed{7} \checkmark$$

check by Gauss elim:

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 7 \end{vmatrix} = \boxed{7} \checkmark$$



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