
Final Exam

Linear Algebra, Dave Bayer, May 11, 2004

Name: Solutions

[1] (5 pts)	[2] (5 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	[6] (6 pts)	[7] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided, and identify all continuations clearly.

[1] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors $(1, 0, 0, 1)$, $(0, 1, 0, 1)$, $(0, 0, 1, 1)$.

answer:

$$\{ (1,0,0,1),\ (-1,2,0,1),\ (-1,-1,3,1) \}$$

work: Eyeball it, need 3 vectors ⊥ to $(1,1,1,-1)$:

How about $\{ (0,0,1,1), (0,2,-1,1), (3,-1,-1,1) \}$? (⊥, ⊥)

Gram Schmidt:

$u_1 = (1,0,0,1)$ $\quad v_1 = (1,0,0,1)$

$u_2 = (0,1,0,1)$ $\quad v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = u_2 - \frac{1}{2} v_1$

$u_3 = (0,0,1,1)$ $\quad \sim 2u_2 - v_1 = (-1,2,0,1)$

$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = u_3 - \frac{1}{2} v_1 - \frac{1}{6} v_2$

$\sim 6u_3 - 3v_1 - v_2 = (-2,-2,6,2)$

$\sim (-1,-1,3,1)$

(same idea as first try)

check: ☑ ⊥ to each other
☑ ⊥ to $(1,1,1,-1)$

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[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data $(x_1, y_1) = (-1, 0)$, $(x_2, y_2) = (0, 0)$, $(x_3, y_3) = (1, 1)$, $(x_4, y_4) = (2, 0)$.

answer:

$$y = \frac{1}{10}x + \frac{2}{10}$$

work:

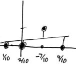
$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Ax = b \Rightarrow A^T A x = A^T b$$

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/10 \\ 2/10 \end{bmatrix}$$

x	y	$\frac{1}{10}x + \frac{2}{10}$	Δ
-1	0	1/10	1/10
0	0	2/10	2/10
1	1	3/10	-7/10
2	0	4/10	4/10



1/10 2/10 -7/10 4/10 sums to 0 ✓

$-1 \cdot \frac{1}{10}$ × $1 \cdot \frac{-7}{10}$ $2 \cdot \frac{4}{10}$ grab at 0, no twist ✓

-1/10 -7/10 8/10

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[3] Find (s,t) so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$ is as close as possible to $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

answer:

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}$$

work:

$$Ax=b \Rightarrow A^TAx = A^Tb$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}$$

check:

$$A\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \\ 2/5 \\ 2/5 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/5 \\ 1/5 \\ 2/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -1/5 \\ -2/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ 3 \end{bmatrix}/5 \perp \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

☑ ☑

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[4] Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D.

answer:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}/2$$

work:

$$\det|A-\lambda I| = \lambda^2 - \text{trace}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda-2)(\lambda-4) = 0$$

$\lambda = 2, 4$

$\lambda = 2$: $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v = (1,-1)$

$\lambda = 4$: $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v = (1,1)$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & \\ & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}/2$$

S←S (S←E E←E E←S

$$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 4 \end{bmatrix}/2$$

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[5] Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 1 & -2 \\ 3 & 1 & -3 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D.

answer:

$$\begin{bmatrix} 2 & 1 & -2 \\ 2 & 1 & -2 \\ 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

work:

$$-\det|A-\lambda I| = \lambda^3 - \underbrace{\operatorname{trace}(A)}_{\substack{2+1+(-3) \\ 0}}\lambda^2 + \underbrace{\operatorname{trace}(\Lambda^2 A)}_{\substack{\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 3 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} \\ 0 \quad\quad 0 \quad\quad -1}}\lambda - \underbrace{\det(A)}_{0,\ \text{singular}} = 0$$

$$= \lambda^3 - \lambda = 0 \quad (\lambda-1)(\lambda+1)\lambda = 0$$

$\lambda = -1, 0, 1$

$\lambda = -1$: $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & -2 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \vec{0}$

$\lambda = 0$: $\begin{bmatrix} 2 & 1 & -2 \\ 2 & 1 & -2 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$

$\lambda = 1$: $\begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & -2 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$

$$\underset{S \leftarrow S}{\begin{bmatrix} 2 & 1 & -2 \\ 2 & 1 & -2 \\ 3 & 1 & -3 \end{bmatrix}} = \underset{S \leftarrow E}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}} \underbrace{\underset{E \leftarrow E}{\begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix}} \underset{E \leftarrow S}{\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}} / 1}_{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}}$$

$$\begin{array}{c|cccc} 1 & 1 & 2 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array}$$

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[6] Let $A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$. Find the matrix exponential e^{At}.

answer:

$$e^{At} = \begin{bmatrix} e^{-t} - te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} + te^{-t} \end{bmatrix}$$

work:

$\det|A - \lambda I| = \lambda^2 - \text{trace}(A)\lambda + \det(A) = \lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\lambda = -1, -1$

$B = A - \lambda I = A + I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ can't find basis v_1, v_2 so $v_1, v_2 \xrightarrow{B} 0$

instead find $v_2 \xrightarrow{B} v_1 \xrightarrow{B} 0$

nearly any v_2 will do in 2x2 case

$v_2 = (0,1)$

$Bv_2 = (1,1) = v_1$

$Bv_1 = 0$ ✓

$(0,1) \xrightarrow{B} (1,1) \xrightarrow{B} (0,0)$ ✓

(v_2, v_1)

$\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

S←S = S←E E←E E←S

$\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ ✓

$e^{At} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

let $a = e^{-t}$, $b = te^{-t}$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\begin{bmatrix} a-b & b \\ -a & a \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{-t} - te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} + te^{-t} \end{bmatrix} = \begin{bmatrix} a-b & b \\ -b & a+b \end{bmatrix}$

checks: $e^{At}\big|_{t=0} = I$ → $t=0$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$\left(\frac{d}{dt} e^{At}\right)\Big|_{t=0} = A$ — $\frac{d}{dt}$ then $t=0$ $(a \to -1 \;\; b \to 1)$ → $\begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$ ✓

forgot sign until this check!
(was using e^t not e^{-t}, oops!)

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[7] Let $A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find the matrix exponential e^{At}.

answer:

$$e^{At} = \begin{bmatrix} e^t + te^t & te^t & -te^t \\ e^t - e^{2t} & e^t & -e^t + e^{2t} \\ e^t + te^t - e^{2t} & te^t & -te^t + e^{2t} \end{bmatrix}$$

work:

$$-\det|A - \lambda I| = \lambda^3 - \underbrace{\text{trace}(A)}_{\substack{2+1+1 \\ 4}}\lambda^2 + \underbrace{\text{trace}(\Lambda^2 A)}_{\substack{\left|\begin{smallmatrix}2&1\\-1&1\end{smallmatrix}\right| + \left|\begin{smallmatrix}2&-1\\0&1\end{smallmatrix}\right| + \left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right| \\ 3 \quad 2 \quad 0 \\ 5}}\lambda - \underbrace{\det(A)}_{\substack{2\left|\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right| - (-1)\left|\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right| \\ 2}} = 0$$

$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$ try $-2 \quad -1 \quad 0 \quad 1 \quad 2$
$\qquad\qquad\qquad\qquad\qquad\qquad -2 \quad 0 \quad 0$

$(\lambda-1)(\lambda-1)(\lambda-2)$

$(\lambda^2 - 2\lambda + 1)(\lambda - 2)$ $\boxed{\lambda = 1, 1, 2}$

$\lambda = 2$: $\begin{bmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$ $v = (0,1,1)$

$\lambda = 1$: $B = A - 1I = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ rank 2, can't find indep v_1, v_2 so $v_1, v_2 \xrightarrow{B} 0$

find instead $v_2 \xrightarrow{B} v_1 \xrightarrow{B} 0$

$B^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$ (most v_2 so $B^2 v_2 = 0$ work, check $Bv_2 \neq 0$.)

but $B\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$, no good.

$B^2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$ $B\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \neq 0$, good. $B\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$ ✓

~~$v_1 =$~~ $v_2 = (0,1,0)$, $v_1 = (1,0,1)$

$$\boxed{\begin{matrix} v_2 & \xrightarrow{B} & v_1 & \xrightarrow{B} & 0 \\ \| & & \| & & \\ (0,1,0) & & (1,0,1) & & \end{matrix}}$$

Continued on ~~back of~~ page 8

Problem: 7 continued

$$\begin{bmatrix}2&1&-1\\-1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}1&1&\\&1&\\&&2\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&-1\\-1&0&1\end{bmatrix}$$

$S \leftarrow S \quad S \leftarrow E \quad E \leftarrow E \quad E \leftarrow S$

$$\begin{bmatrix}2&1&-1\\1&1&-1\\-2&0&2\end{bmatrix}$$

1	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	1	1	0
0	1	0	0	1

$$e^{At} = \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}a&b&\\&a&\\&&c\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&-1\\-1&0&1\end{bmatrix}$$

where $a = e^t$, $b = te^t$, $c = e^{2t}$

$$\begin{bmatrix}a+b&b&-b\\a&a&-a\\-c&0&c\end{bmatrix}$$

$t=0$: $a=1$, $b=0$, $c=1$

$\frac{d}{dt}$ then $t=0$: $a=1$, $b=1$, $c=2$

$$\begin{bmatrix}a+b&b&-b\\a-c&a&-a+c\\a+b-c&b&-b+c\end{bmatrix}$$

$t=0$: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ✓ I

$\frac{d}{dt}$ then $t=0$: $\begin{bmatrix}2&1&-1\\-1&1&1\\0&1&1\end{bmatrix}$ ✓ A

$$\begin{bmatrix}e^t+te^t & te^t & -te^t\\ e^t-e^{2t} & e^t & -e^t+e^{2t}\\ e^t+te^t-e^{2t} & te^t & -te^t+e^{2t}\end{bmatrix}$$

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