Comb ① Thur 2/7/02

Redo paths problem w/
- count, not weights (simpler)
- sensible notation

$$X = \begin{bmatrix} x_{11}, x_{12} \\ \vdots \\ x_{n1}\ x_{n2} \end{bmatrix}$$ matrix whose rows are <u>starts</u> of paths

$$Y = \begin{bmatrix} y_{11}\ y_{12} \\ \vdots \\ y_{n1}\ y_{n2} \end{bmatrix}$$ matrix " " " <u>ends</u> of paths

and $x_i = (x_{i1}, x_{i2})$, $y_i = (y_{i1}, y_{i2})$

Given $\pi \in S_n$, Y_π is Y w/rows permuted by π:

$$Y_\pi = \begin{bmatrix} y_{\pi(1)1} & y_{\pi(1)2} \\ & \vdots & \\ y_{\pi(n)1} & y_{\pi(n)2} \end{bmatrix}$$

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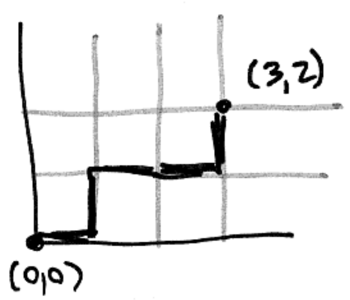
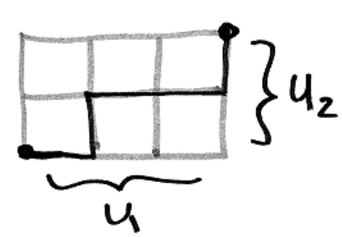
paths go <u>over</u> $\rightarrow$ or <u>up</u> $\uparrow$ $((1,0)$ or $(0,1))$



n-path L is set of n paths
$\{ L_i$ from x_i to $y_i \}$ $i = 1..n$
~~whose~~ its <u>type</u> is (X, Y)

$h(u)$, $u = (u_1, u_2)$, is # paths



$\Leftrightarrow$ out of $u_1 + u_2$ steps,
where are the u_1 horizontal steps?

$$h(u) = \begin{cases} \binom{u_1+u_2}{u_1} & u \geq 0 \\ 0 & \text{else} \end{cases}$$

\# n-paths L of type (X,Y) is therefore

$$\boxed{\prod_{i=1}^{n} h(y_i - x_i)}$$

\# n-paths L of type (X, Y_π) is therefore

$$\boxed{\prod_{i=1}^{n} h(y_{\pi(i)} - x_i)}$$

<u>Theorem</u> # <u>nonintersecting</u> n-paths of type (X,Y) is

$$\det \begin{bmatrix} h(y_1-x_1) & h(y_2-x_1) & \cdots & h(y_n-x_1) \\ h(y_1-x_2) & h(y_2-x_2) & \cdots & h(y_n-x_2) \\ \vdots & & & \vdots \\ h(y_1-x_n) & h(y_2-x_n) & \cdots & h(y_n-x_n) \end{bmatrix}$$

$$= \sum_{\pi \in S_n} (-1)^{\pi} \prod_{i=1}^{n} h(y_{\pi(i)} - x_i)$$

proof: Let $\mathcal{Q}_\pi$ = set of n-paths of type (X, Y_π),

so $$\# \mathcal{Q}_\pi = \prod_{i=1}^{n} h(y_{\pi(i)} - x_i)$$

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let $\mathcal{Q} = \bigcup_{\pi \in S_n} \mathcal{Q}_\pi$

we can pair n-paths in $\mathcal{Q}$ that intersect,

L in $\mathcal{Q}_\pi$ with L' in $\mathcal{Q}_\sigma$,

so $(-1)^\pi + (-1)^\sigma = 0$, canceling all terms in det except those counting nonintersecting paths, in $\mathcal{Q}_{id}$

Pairing rule: Lex order $(1,1) < (1,2) < \ldots < (1,n) < (2,1) < (2,2) < \ldots$

Given $L \in \mathcal{Q}_\pi$, intersecting, choose least (i,j) so L_i and L_j intersect, and least vertex $z = (z_1, z_2)$ where they intersect. Swap the continuations past z, get L'.

- This is an <u>involution</u>. Rule locates same crossing, swaps back.

- If L has type π, $\pi = (\pi(1), \ldots \pi(i), \ldots \pi(j), \ldots \pi(n))$
  then L' " " σ, $\sigma = (\pi(1), \ldots \pi(j), \ldots \pi(i), \ldots \pi(n))$

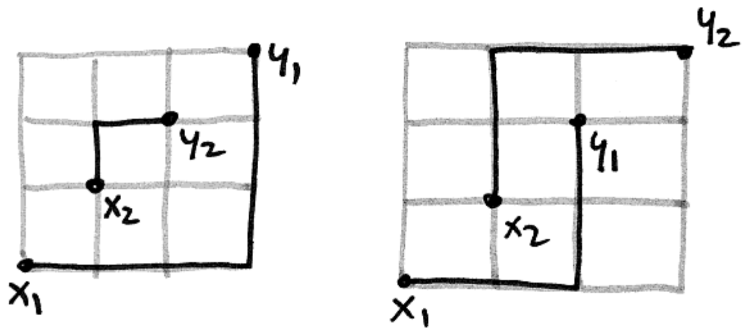
  pair swap

  so $\sigma = (i\,j)\pi$ and $(-1)^\sigma = (-1)(-1)^\pi$

- If there are any nonintersecting paths in $\mathcal{Q}_{id}$, there are none in any other $\mathcal{Q}_\pi$, so det cancels out, as wanted. //

needed hypothesis

(Sure looks like if $\mathcal{Q}_\pi$ is feasible for $\pi \neq id$, and $(-1)^\pi = -1$, then det gets sign wrong. Does Stanley say this?)

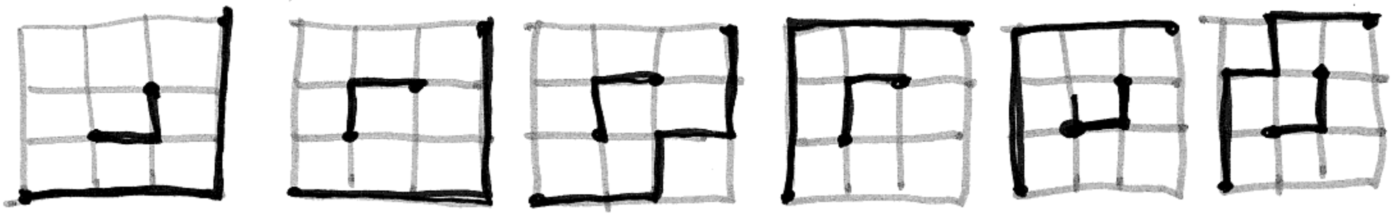


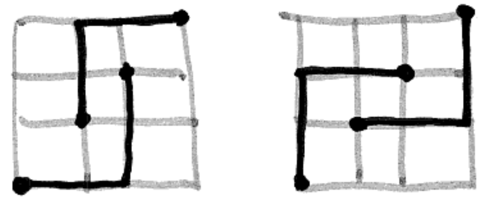
Huh? Looks like both $\mathcal{Q}_{id}$ and $\mathcal{Q}_{(12)}$ have nonintersecting paths. What gives?

$$X = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$$

$$\det \begin{vmatrix} h(y_1 - x_1) & h(y_2 - x_1) \\ h(y_1 - x_2) & h(y_2 - x_2) \end{vmatrix} = \det \begin{vmatrix} h(3,3) & h(2,2) \\ h(2,2) & h(1,1) \end{vmatrix}$$

$$= \det \begin{vmatrix} \binom{6}{3} & \binom{4}{2} \\ \binom{4}{2} & \binom{2}{1} \end{vmatrix} = \det \begin{vmatrix} 20 & 6 \\ 6 & 2 \end{vmatrix} = 40 - 36 = 4$$



$-$  $= 4$

So what condition insures that only $\mathcal{A}_{id}$ has nonintersecting paths?

Ahh! 2.7.1 Stanley states <u>exactly</u> this as hypothesis, only $\mathcal{B}(X,Y)$ is nonempty, $\mathcal{B}(X, Y_\pi) = \emptyset$ if $\pi \neq id$, where $\mathcal{B}(X,Y)$ is set of nonintersecting n-paths, type (X,Y).

$\boxed{\text{chapter 3, Posets}}$

(To avoid nerve damage, we'll pick up most defs as we need them)

P is a poset (<u>par</u>tially <u>o</u>rdered <u>set</u>)

$\iff$ P is a set w/ relation $\leq$

1) $x \leq x \quad \forall x$
2) $x \leq y,\ y \leq x \Rightarrow x = y, \quad \forall x, y$
3) $x \leq y,\ y \leq z \Rightarrow x \leq z, \quad \forall x, y, z$

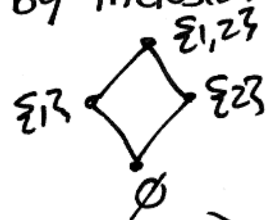
Examples

$\mathbf{n} = \{1..n\}$, usual order $1<2<\ldots<n$

$[n]$ chain

$B_n = 2^{[n]}$, subsets of $[n]$ by inclusion

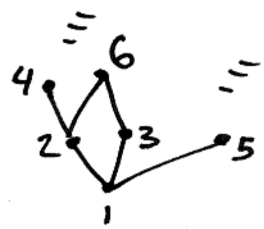
binary poset



(looks like hypercube so $(0,..0)$ to $(1,..1)$ is vertical)

$D_n = \{n \in \mathbb{P}\}$ $\quad \mathbb{P} = 1,2,3\ldots$

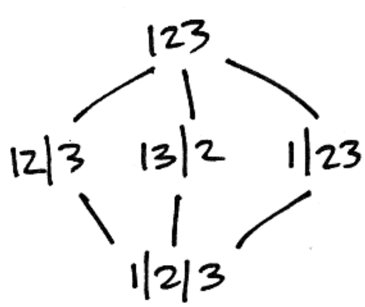
by divisibility



Π_n = partitions of $[n]$

by refinement (finer is smaller)

partions of $[3]$ are $1|2|3$, $12|3$, $13|2$, $1|23$, 123



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$L_n(q)$ = all subspaces of $\mathbb{F}_q^n$, under inclusion

'q-analogue' of binary poset B_n

<u>chain</u> linearly ordered subset of P

$x_1 < \ldots < x_n$

<u>multichain</u> " " w/ repeats allowed

$x_1 \leq x_2 \leq \ldots \leq x_n$

(If you're thinking <u>monomials</u> you'd be right...)

<u>antichain</u> (clutter) totally incomparable ~~set~~ subset

<u>order ideal</u> $I \subset P$

if $x \in I$ and $y \leq x$ then $y \in I$

(generates "down" unlike monomial ideals, which generate "up")

<u>dual order ideal</u> (filter) $I \subset P$, if $x \in I$ and $y \geq x$ then $y \in I$

(behaves like ring ideals)

If P finite, $\{\text{antichains } A\} \overset{1:1}{\longleftrightarrow} \{\text{order ideals } I\}$

$$I(A) = \{x \in P \mid x \leq y \text{ some } y \in A\}$$

$$A(I) = \{x \in I \mid \underbrace{x < y \Rightarrow y \notin I}_{\text{maximal elements of } I}\}$$

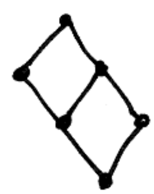
//

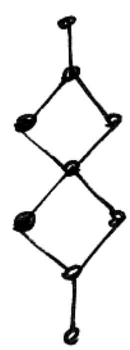
Fundamental Theorem of Finite Distributive Lattices

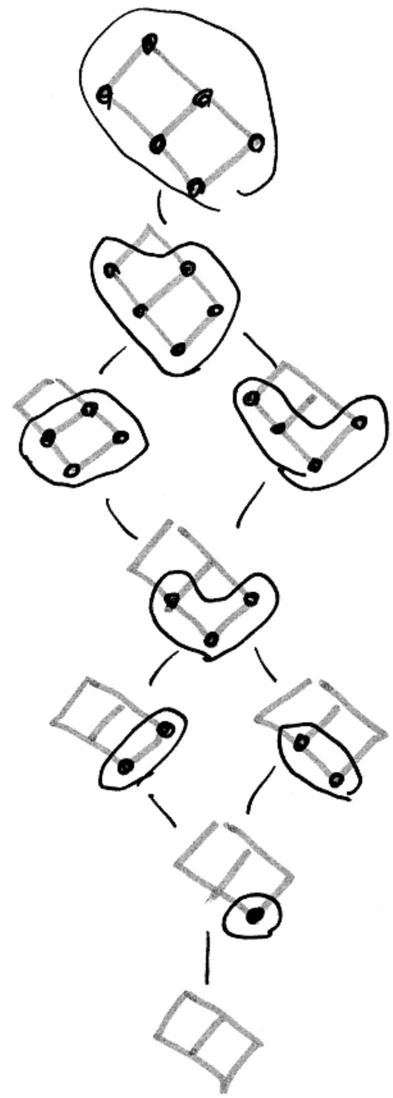
If L is a finite distributive lattice, then $\exists!$ finite poset P so $L \cong J(P)$

what does this theorem mean? (we're hopping ahead to a result)

$\boxed{J(P)}$ = poset of order ideals of P, under inclusion
on

$P =$ 

$J(P) =$ 

$J(P) =$ 

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lattice $\quad x, y \in P \quad z$ upper bound if $x \le z, y \le z$

least " " ⊛

denoted $x \vee y$, "x join y", "x sup y"

same for greatest lower bounds,

$x \wedge y$, "x meet y" "x inf y"

lattice is poset P so every pair x, y has $x \vee y$, $x \wedge y$

$\vee, \wedge$
- associative
- commutative
- idempotent $\quad x \vee x = x \wedge x = x$
- absorbtion laws: $\underbrace{x \wedge (x \vee y) = x \vee (x \wedge y)} = x$
- $x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \le y$

All finite lattices have $\hat{0}$ conventional name for least elem

$\hat{1}$ " " greatest

---

prop P $\underbrace{\text{finite meet-semilattice}}_{\text{meets work, joins?}}$ w/ $\hat{1} \Rightarrow$ lattice

proof need to show $x, y \in P$ has l.u.b. $x \vee y$:

$\hat{1} \in S = \{z \in P \mid z \ge x, z \ge y\} \ne \emptyset$

define $\wedge S = s_1 \wedge s_2 \wedge .. \wedge s_k$, $S = \{s_1, .., s_k\}$

then $x \vee y = \wedge S$ //

⊛ think about what least means

(we skip <u>semimodular</u> for the moment)

<u>Distributive</u> <u>lattice</u> satisfies $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(either implies the other)

~~The partition lattice not distributive~~

n, B_n, D_n distributive lattices

Π_n, $L_n(q)$ lattices but <u>not</u> distributive

$J(P)$ distributive lattice (hence the Thm)

---

think of $\vee$, $\wedge$ as ordinary union $\cup$ and intersection $\cap$

on distinguished subsets $T \subset 2^S$

T stable under $\cup, \cap$

get subposet of $B_{|S|}$ under inclusion, dist. lattice.

---

order ideals $J(P)$ of P enough flexibility to be universal

<u>proof</u> of theorem

$x \in L$ is join-irreducible if not join of strictly smaller elems

order ideal $I \subseteq P$ join-irred in $J(P)$

$\iff$ principle order ideal $I = (x)$ some $x \in P$.

$\Rightarrow \{$join-irreds of $J(P)\} \approx P$

as induced subposet of $J(P)$

$\Rightarrow$ $\boxed{J(P) \cong J(P) \iff P \cong Q}$

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so idea of proof is clear, construct P backwards from $J(P) \cong L$ distributive lattice, show it all works

def P is ~~set~~ induced subposet of join-irreds of L

want to show $J(P) \cong L$

Given $x \in L$, define ~~principal order ideal (x)~~

$$I_x = \{ y \in P \mid y \le x \}$$

not L!

$I_x \in J(P)$ so $\phi : L \xrightarrow{x \mapsto I_x} J(P)$

is order-preserving, meet-preserving injection

---

need to show ϕ is surjective:

given $I \in J(P)$, let $x = \bigvee I$

least upper bound in L of I

want to show that $I = I_x$, in image of ϕ

check: $I \subseteq I_x$ ☑

suppose $z \in I_x$, want to show $z \in I$:

$$\bigvee \{ y \mid y \in I \} = \bigvee \{ y \mid y \in I_x \} \qquad \underbrace{\boxed{\bigvee I = \bigvee I_x}}_{\text{in } L}$$

apply $\wedge z$: $$\bigvee \{ y \wedge z \mid y \in I \} = \underbrace{\bigvee \{ y \wedge z \mid y \in I_x \}}_{\text{just } z}$$

$\Rightarrow$ some $y \in I$ satisfies $y \wedge z = z$, i.e. $z \le y \Rightarrow z \in I$

(cool)

<u>summary</u>

given L, $P \subseteq L$ is induced subposet
of $x \in L$ join-irred
(not join of strictly smaller pairs)

now,

$$\begin{tikzcd} J(P) \arrow[rr, bend left, "I \mapsto x = \bigvee I"] & & L \arrow[ll, bend left, "I_x \leftarrow\!\shortmid\, x"] \end{tikzcd}$$

$I_x = \{y \in P \mid y \leq x\}$

want to show $\circlearrowright$ $I = I_x$

$I \subseteq I_x$ is clear

$I \supseteq I_x$? $z \in I_x$, want $z \in I$:

$$\bigvee I = \bigvee I_x = x$$

apply $\wedge z$: (use distributive law)

$$\underbrace{\bigvee \{y \wedge z \mid y \in I\}} = \underbrace{\bigvee \{y \wedge z \mid y \in I_x\}}_{z}$$

some $y \in I$ satisfies $y \wedge z = z$ join-irred
so $z \leq y$, so
$z \in I$