Tues (1) 5 Feb 02
Comb class

Recall: $\pi \in S_n$ $G(\pi) = \{(i, \pi(i))\} \subset [n] \times [n]$

$B \subseteq [n] \times [n]$ "board" of forbidden positions

$N_j = \#\{\pi \in S_n \mid j = \#(B \cap G(\pi))\}$

perms w/ j forbidden positions

r_k = # non-attacking positions, k rooks on B

Theorem $\boxed{\sum_{j=0}^{n} N_j x^j = \sum_{k=0}^{n} r_k (n-k)!\,(x-1)^k}$

second proof x positive int

$\boxed{\sum_{j=0}^{n} N_j x^j}$ # ways place n non-attacking rooks on $[n] \times [n]$,
then label each rook on B from $\{1, .., x\}$
label remaining rooks 1

$\boxed{\sum_{k=0}^{n} r_k (n-k)!\,(x-1)^k}$ # ways
place k non-attacking rooks on B
label from $\{2, .., x\}$
extend to n nonattacking rooks on $[n] \times [n]$
label remaining rooks 1

polys equal for all x $\Leftrightarrow$ equal as polys

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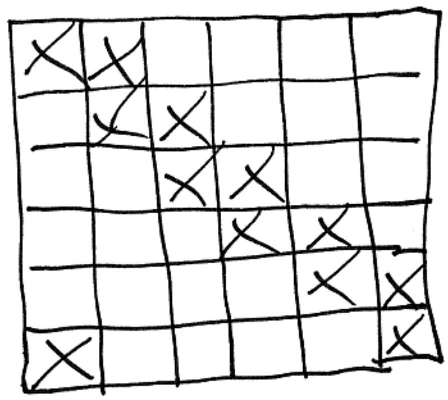
Application, problème des ménages long history unsatisfactory solus

$\# \{ \pi \in S_n \mid \pi(i) \neq i, i+1 \ (\text{mod } n) \text{ for } i = 1..n \}$



r_k = # ways choosing k nonadjacent points from 2n in a circle

$B = \{ (i,i), (i, \underbrace{i+1}_{\text{mod } n}) \mid i = 1..n \}$

lemma k from m in circle $= \frac{m}{m-k} \binom{m-k}{k}$

first proof $f(m,k)$ = desired #
$g(m,k)$ = ... and mark one point not chosen

$\boxed{g(m,k) = (m-k) f(m,k)}$ (explains denom)

mark a point, m ways
then choose k nonadjacent from linear array m-1
{ place m-1-k points 'not chosen' in a line
m-k slots in between
choose k of these for points 'chosen'

$\Rightarrow \boxed{g(m,k) = m \binom{m-k}{k}}$

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2nd proof label points 1..m cyclic order
color k red, nonadjacent

1 isn't red: place m-k uncolored pts in circle
label one '1'
$\underbrace{\text{insert } k \text{ red points into } m-k \text{ gaps}}_{\boxed{\binom{m-k}{k} \text{ ways}}}$

1 is red: place m-k+1 uncolored pts in circle
color one red and label '1'
$\underbrace{\text{insert } k-1 \text{ red points into } m-k-1 \text{ gaps}}_{\boxed{\binom{m-k-1}{k-1} \text{ ways}}}$

$$\binom{m-k}{k} + \binom{m-k+1}{k-1} = \left[1 + \frac{k}{m-k}\right]\binom{m-k}{k} = \boxed{\frac{m}{m-k}\binom{m-k}{k}}$$

corollary

$$\boxed{\begin{aligned} \sum N_j x^j &= \sum_{k=0}^{\infty} \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!\,(x-1)^k \\ N_0 &= \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!\,(-1)^k \end{aligned}}$$

hard formula broken into easy steps

finish chapter by skipping 2.4, 2.5
doing 2.6, 2.7

want <u>combinatorial</u> proof (bijection of sets) of

$$\boxed{f_=(\emptyset) = \sum_Y (-1)^{|Y|} f_\geq(Y)}$$

where $Y \subseteq S$, properties we wish to forbid on objects $x \in A$

rewrite as

$$\boxed{f_=(\emptyset) + \sum_{|Y| \text{ odd}} f_\geq(Y) = \sum_{|Y| \text{ even}} f_\geq(Y)} \quad (*)$$

look at triples (x, Y, Z)
$x \in A$ object
$Y, Z \subseteq S$ sets of properties
$\boxed{Y \subseteq Z, \text{ and } x \text{ has exactly properties } Z}$

we order properties in S, and define <u>involution</u>

$$\sigma(x,Y,Z) = \begin{cases} (x, Y-i, Z) & \text{if } Y \neq \emptyset \\ & \min Y = \min Z = i \\ (x, Y+i, Z) & \text{if } Z \neq \emptyset \\ & \min Z = i < \min Y \\ & \text{(yada yada if } Y \neq \emptyset) \\ (x, \emptyset, \emptyset) & \text{if } Z = \emptyset \end{cases}$$

involution, with $(x, \emptyset, \emptyset)$ as fixed points,
appearing on both sides of $(*)$
all other ~~pairs~~ triples on one side or other,
involution pairs them.

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restate: $X = X^+ \cup X^-$

τ involution on X that either

or { • pairs elements in X^+, X^-
• fixes elems of X^+

Then $\boxed{\#(\mathrm{Fix}\,\tau) = \sum_X \omega(x)}$ $\begin{cases} +1, & x \in X^+ \\ -1, & x \in X^- \end{cases}$

(another way to understand "triples" view of inclusion/exclusion)

Another set $\tilde{X} = \tilde{X}^+ \cup \tilde{X}^-$

$\tilde{\tau}$ on $\tilde{X}$ like above

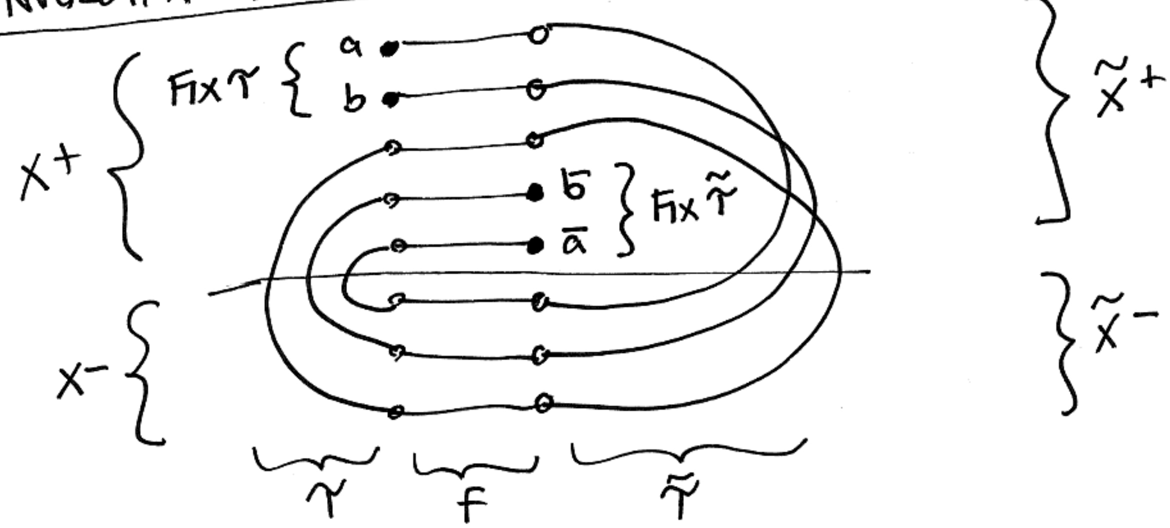
f: sign-preserving bijection $X \to \tilde{X}$

$\Rightarrow \#(\mathrm{Fix}\,\tau) = \#(\mathrm{Fix}\,\tilde{\tau})$

but can we construct canonical bijection

$g: (\mathrm{Fix}\,\tau) \to (\mathrm{Fix}\,\tilde{\tau})$?

INVOLUTION PRINCIPLE



every path either a loop or has two ends example

Ψ : $\pi \in S_n$ so $\pi(1) = 1$

$\tilde{\Psi}$: $\pi \in S_n$ exactly one cycle (n-cycle)

$\#\Psi = \#\tilde{\Psi} = (n-1)!$

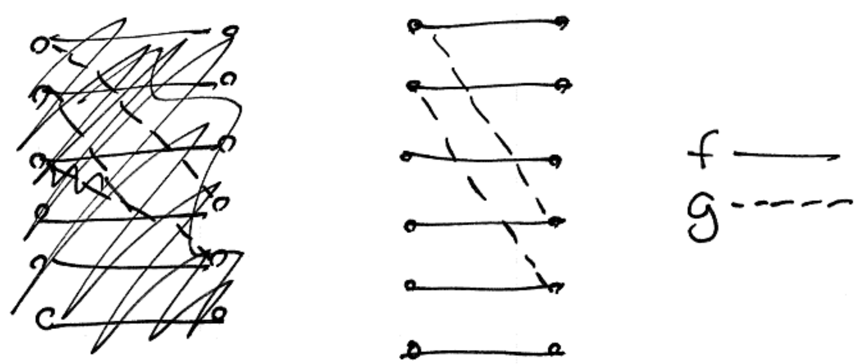
~~define~~ related question $\Psi \subseteq X$, $\tilde{\Psi} \subseteq \tilde{X}$

have bijections $f : X \to \tilde{X}$

$g : \Psi \to \tilde{\Psi}$

want bijection $h : X - \Psi \to \tilde{X} - \tilde{\Psi}$

draw picture



h matches endpoints of resulting paths

$f : X \to \tilde{X}$ identity on $X, \tilde{X} = S_n$

$g : \Psi \to \tilde{\Psi}$

$1 a_2 \cdots a_n \mapsto (1\, a_2 \cdots a_n)$

h: write in cycle form.
if one cycle, treat as perm,
do it again.

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$$\begin{array}{llll}
\Psi\left\{\begin{array}{l}123\\132\end{array}\right. & \begin{array}{l}\leftarrow\\\leftarrow\end{array} & \begin{array}{l}(1)(2)(3)\\(1)(23)\end{array} & \\
213 & \text{—} & (12)(3) & \\
231 & \rightarrow & (123) & \\
312 & \rightarrow & (132) & \left.\right\}\tilde{\Psi}\\
321 & \text{—} & (13)(2) &
\end{array}$$

} example for $n=3$

scaffolding. Can we more descriptively understand h?

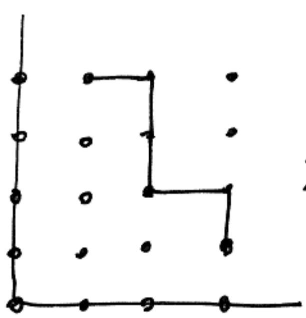
$\Psi, \tilde{\Psi}$ disjoint for $n \geq 2$

$$h(\pi) = \begin{cases} \pi, & \pi \notin \tilde{\Psi} \\ g^{-1}(\pi), & \pi \in \tilde{\Psi} \end{cases}$$

2.7 determinants

lattice path in plane $\quad \rightarrow x$, $\downarrow y$ $\quad$ right or down

$L = (v_1, \ldots, v_k) \qquad v_{i+1} - v_i = (1,0)$ or $(0,-1)$



$x_2 x_4$

n-path $\vec{L} = (L_1, \ldots, L_n)$

type $(\alpha, \beta, \gamma, \delta)$

if L_i goes (β_i, γ_i) to (α_i, δ_i)

$\alpha \geq \beta, \quad \gamma \geq \delta$

(whacky sign conventions, but hey!)

$\vec{L}$ nonintersecting if L_i's disjoint

weight of horizontal step (i,j) to $(i+1,j)$ is x_j

$\leftarrow$ "height"

let $\pi \in S_n$ act on n-tuples by shuffling coords

$$\pi(\alpha_1, \ldots \alpha_n) = (\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)})$$

$\mathcal{A}$ = set of all n-paths type $(\alpha, \beta, \gamma, \delta)$

$A = A(\alpha, \beta, \gamma, \delta)$ sum of their weights

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if L_i path (β_i, γ_i) to (α_i, δ_i)

weight $x_{k_1} \cdots x_{k_m}$ $\quad m = \alpha_i - \beta_i$

$$\gamma_i \geq k_1 \geq \ldots \geq k_m \geq \delta_i$$

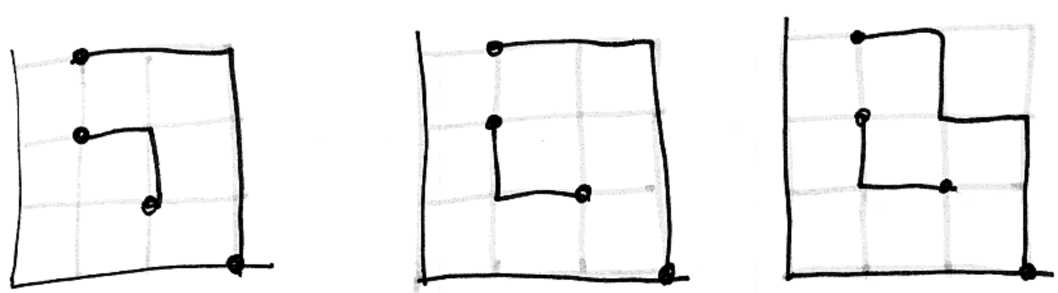
so define $h(m, \gamma_i, \delta_i) = \sum\limits_{\text{all seqs}} x_{k_1} \cdots x_{k_m}$

then $A(\alpha, \beta, \gamma, \delta) = \prod\limits_{i=1}^{n} h(\alpha_i - \beta_i, \gamma_i, \delta_i)$

$\mathcal{B}$ = set of all nonintersecting n-paths $(\alpha, \beta, \gamma, \delta)$

$\mathcal{B} = B(\alpha, \beta, \gamma, \delta)$ sum of their weights

example paths from $\boxed{\begin{matrix}(1,2)\\(1,3)\end{matrix}}$ to $\boxed{\begin{matrix}(2,1)\\(3,0)\end{matrix}}$



$x_2 x_3^2$ $\qquad$ $x_1 x_3^2$ $\qquad$ $x_1 x_2 x_3$

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Theorem $\mathcal{B} = \det\left[\underbrace{h(\alpha_j - \beta_i, \gamma_i, \delta_j)}_{0 \text{ when no such sequences}}\right]_1^n$

proof ~~Beta~~ $\det = \sum_{\pi \in S_n} (-1)^\pi A(\pi(\alpha), \beta, \gamma, \pi(\delta))$

unpermuted start to permuted end

define $\mathcal{Q}_\pi = \mathcal{Q}(\pi(\alpha), \beta, \gamma, \pi(\delta))$

construct bijection τ $\left(\bigcup_{\pi \in S_n} \mathcal{Q}_\pi \right) - \mathcal{B}$ to self

- involution
- weight-preserving
- flips signs of permutations

reveals that all terms cancel out in det except those of $\mathcal{B}$

(note $\mathcal{B} \subset \mathcal{Q}$, $\mathcal{Q}_\pi$ everything crosses unless $\pi = id$)

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many ways to construct τ

pick least i so L_i intersects ~~k~~ some L_k, $k > i$

least $j > i$ so L_i, L_j intersect

"least intersecting pair (L_i, L_j) in lex sense"

construct L_i^*, L_j^* by flipping L_i, L_j at ~~last~~ first intersection.

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pretty slick

important apps in thy symmetric fns, but we look at simple case:

$r, s \quad S \subset [0,r] \times [0,s]$

how many lattice paths $(0,r)$ to $(s,0)$ avoiding S?

$\# = f(r,s,S)$

$$S = \{(a_1,b_1),\dots,(a_k,b_k)\}$$

n-paths

from $(\beta = (0,a_1,\dots,a_k),\ \gamma = (r,b_1,\dots,b_k))$

to $(\alpha = (s,a_1,\dots,a_k),\ \delta = (0,b_1,\dots,b_k))$

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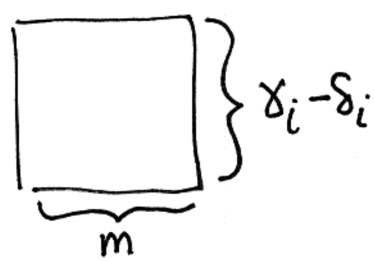
then (setting all $x_j = 1$)

$$f(r,s,S) = B(\alpha,\beta,\gamma,\delta)$$

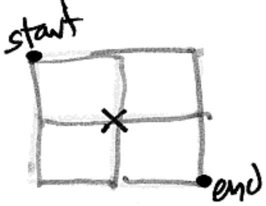
$h(m,\gamma_i,\delta_i)\big|_{x_j=1}$ is just # paths



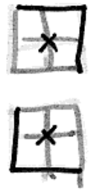
$$= \binom{\gamma_i - \delta_i + m}{m}$$

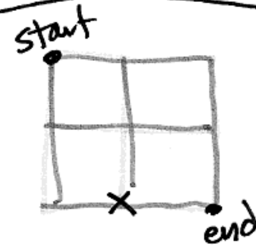
let's find simplest example where this is interesting?

$r=s=1$
$S=\{(0,0)\}$
$\#=1$

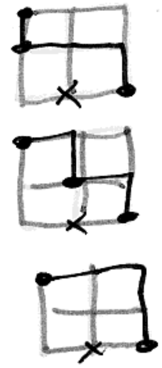


$r=s=2$
$S=\{(1,1)\}$
$\#=2$





$r=s=2$
$S=\{(1,0)\}$
$\#=3$



~~det = | #paths (0,0) to (0,2) (0,2) to (2,0) #paths (2,0) to (1,0) #paths (1,0) to (1,0)~~

$$\det = \begin{vmatrix} \#\text{paths } (0,2) \text{ to } (2,0) & \#\text{paths } (0,2) \text{ to } (1,0) \\ \#\text{paths } (1,0) \text{ to } (2,0) & \#\text{paths } (1,0) \text{ to } (1,0) \end{vmatrix}$$

$$= \begin{vmatrix} \binom{4}{2} & \binom{3}{1} \\ \binom{1}{1} & \binom{0}{0} \end{vmatrix} = \begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix} = 3 \quad \checkmark$$

$\binom{\text{total steps}}{\text{horiz steps}}$

can rearrange this problem to be prev matrix, inc/excl

(in general, not matrix of form $\begin{vmatrix} * & * & * \\ 1 & * & * \\ 0 & 1 & * \end{vmatrix}$)

let's illustrate involution τ for this example: