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comb

[start w/ pages 8-10, Tues 19 Feb 02]

Note Stanley's convention

P poset, $\hat{P}$ has $\hat{0}, \hat{1}$ adjoined (needed or not!)

so $\tilde{\chi}_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$

this is same as us, because $(\hat{0}, \hat{1})$ is P

<u>geometric realization</u> $X = |\Delta|$ of Δ

$\tilde{\chi}(X) = \sum_i (-1)^i \tilde{h}_i(X, \mathbb{Z})$

(compute any way you like singular homology of X
simplicial " " Δ)

so $\mu_{\hat{P}}(\hat{0}, \hat{1})$ depends only on $|X|$ not X

("homotopy equivalence of posets" sneaking in...)

<u>finite</u> <u>regular</u> <u>cell</u> <u>complex</u> Γ = union nonempty
pairwise disjoint

open cells $\sigma_i \subset \mathbb{R}^N$ so

(a) $(\bar{\sigma}_i, \bar{\sigma}_i - \sigma_i) \cong (\mathbb{B}^n, \mathbb{S}^{n-1})$ some $n = n(i)$

(b) each $\bar{\sigma}_i - \sigma_i$ is union of σ_j's

[<u>Cellular homology</u> generalizes <u>simplicial homology</u>

need sign for $\sigma_j \subset \sigma_i$ facet, sign not obvious

any two "good" sign rules equivalent

(see Bruns, Herzog e.g.)]

$|\Gamma|$ realization of Γ

$sd(\Gamma)$ simplicial complex of chains of cells $\sigma_{i_1} \subset \sigma_{i_2} \subset \ldots \sigma_{i_\ell}$ of Γ

first barycentric subdivision

$P(\Gamma)$ facet poset of cells of Γ (don't include empty face $\emptyset$)

then

$$\boxed{sd(\Gamma) = \Delta(P(\Gamma))}$$

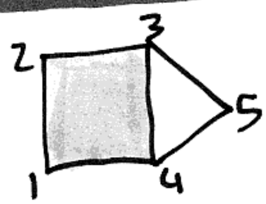
so

$$\boxed{\mu_{\hat{P}}(\hat{0},\hat{1}) = \tilde{\chi}(|\Gamma|)}$$

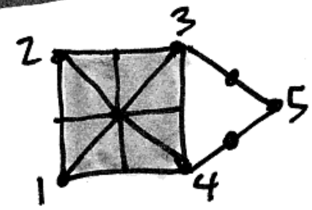
$-$ empty $= -1$
$+\ V = 5$
$-\ e = -6$
$+\ F = 1$

$$\boxed{\tilde{\chi} = -1}$$

example



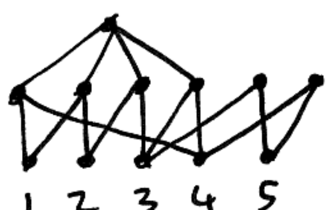
Γ



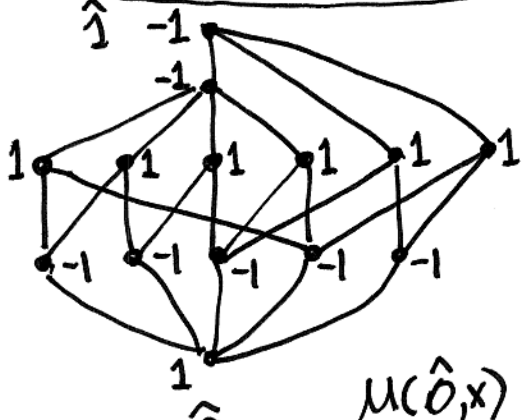
$$\underbrace{sd(\Gamma) = \Delta(P(\Gamma))}$$

$\left(\begin{array}{l} h_*(S^1) = 0 \ \ * \neq 1 \\ h_1(S^1) = 1 \end{array} \right)$

$P(\Gamma)$



$\left\} \begin{array}{l} \text{4 maximal chains length 1} \\ \text{8 " " " 2} \end{array} \right.$



$\hat{P}$:

recursively compute $\mu_{\hat{P}}(\hat{0},\hat{1})$

$$\mu_{xy} = -\sum_{x \leq z < y} \mu_{xz}$$

$$\Rightarrow \boxed{\mu(\hat{0},\hat{1}) = -1}$$

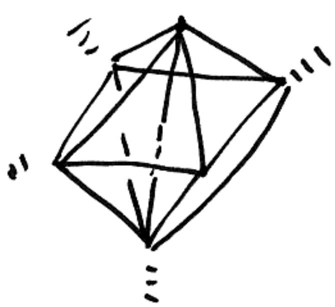
$\mu(\hat{0},x)$ for $x \in \hat{P}$

what about other $\mu_{\hat{P}}(x,y)$?

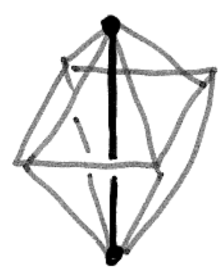
<u>link</u> $\mathrm{lk}\, F$ of a face $F \in \Delta$, Δ simplicial complex, defined by

$$\mathrm{lk}\, F = \{ G \setminus F \mid F \subseteq G \in \Delta \}$$ (restated slightly from Stanley)

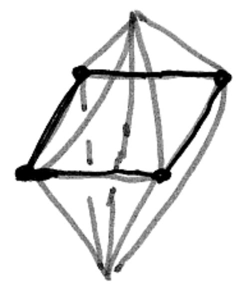
in closed <u>manifold</u>, $\left.\begin{array}{l} \dim F = i \\ \dim \Delta = n \end{array}\right\}$ ~~dim~~ $\mathrm{lk}\, F \cong S^{n-i-1}$



part of Δ, dim 3

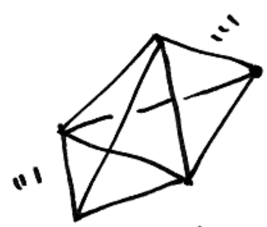


F, dim 1



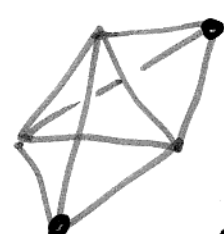
$\mathrm{lk}\, F \cong S^1$ (1-sphere)

simplicial manifold, PL category, becomes our "definition" of a manifold (WARNING: higher dims, wild triangulations of manifolds this fails) <u>but true at level of homology</u>



part of Δ, dim 3

F, dim 2



$\mathrm{lk}\, F \cong S^0$ (2 points)

Γ finite regular cell complex, $|\Gamma| =$ manifold

$$\mu_{\hat{P}}(x,y) = \begin{cases} 0, & x \neq \hat{0},\ y = \hat{1}, \quad x \text{ lies on boundary of } |\Gamma| \ \boxtimes \\ \tilde{\chi}(\Gamma), & x = \hat{0},\ y = \hat{1} \quad \boxtimes \\ (-1)^{\ell(x,y)} & \text{otherwise} \end{cases}$$

we use lk F to express $\mu_{\hat{P}}(x,y)$:

given $x<y$ choose saturated chains in P (not $\hat{P}$)

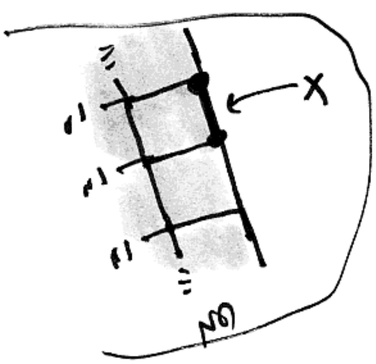
$$\boxed{\underbrace{x_1}_{\text{min}} < x_2 < \dots < x_r = x} < \boxed{y = y_1 < \dots < \underbrace{y_s}_{\text{max}}}$$

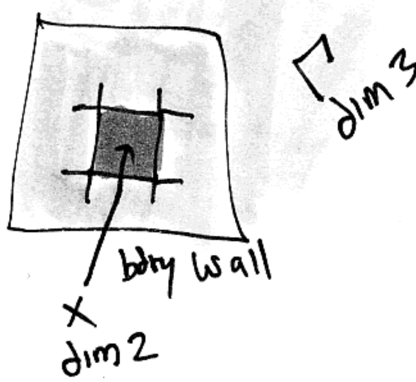
then $\underbrace{\Delta((x,y))}_{\text{our notation so far}} = \mathrm{lk}\, F$ in $\Delta(P)$

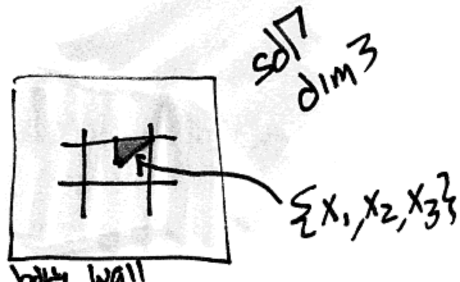
$F = \{x_1, \dots, x_r, y_1, \dots, y_s\}$

(so the family of spaces $\Delta((x,y))$ all related, as links of $\Delta(P)$)

Now, x lies on boundary of Γ







choose $x_1 < x_2 < x_3 = x$

still boundary cell

$\Rightarrow \mathrm{lk}\, F \cong \mathbb{B}^i$ some i

$$\Rightarrow \mu_{\hat{P}}(x,\hat{1}) = \tilde{\chi}(\Delta((x,\hat{1})))$$
$$= \tilde{\chi}(\mathrm{lk}\, F) = 0$$

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remaining case: $\mu_{\hat{P}}(x,y) = (-1)^{\ell(x,y)}$

$x \neq \hat{0}$ $y \neq \hat{1}$ or x not on boundary

How to think of this: $\hat{1}$ has "boundary" Γ

any other y has boundary $\cong S^i$ some i

lk F is usual link of x in boundary of y

so $\tilde{\chi}(\text{lk } F)$ is ± 1, depends on dim of link

homology of a sphere

$\Longrightarrow$ <u>Def</u> P (<u>with</u> $\hat{0}, \hat{1}$) <u>semi-Eulerian</u>

if $\mu_P(x,y) = (-1)^{\ell(x,y)}$ for $(x,y) \neq (\hat{0},\hat{1})$

<u>Eulerian</u>

if also $\mu_P(\hat{0},\hat{1}) = (-1)^{\ell(\hat{0},\hat{1})}$

$|\Gamma|$ manifold w/o boundary $\Rightarrow$ $\hat{P}(\Gamma)$ semi-Eulerian

$|\Gamma|$ sphere $\Rightarrow$ " Eulerian

Example B_n Eulerian $\left(B_n \cong 2^n \quad S,T \subset n \quad \mu(T,S) = (-1)^{|S-T|} \right)$

recall

indeed $B_n = \hat{P}(\Gamma)$, Γ boundary complex of $\underbrace{(n-1)}_{\text{dim}}$-simplex

(subsets of n $\Longleftrightarrow$ faces of simplex) aside: upper bound conjecture for spheres

Given any simplicial sphere, d vertices S^{n-1},

pick d generic points on rational normal curve

$$\gamma(t) = (t, t^2, \ldots, t^n) \hookrightarrow \mathbb{R}^n$$

and take $C_{d,n}$ = boundary of convex hull of

$$\{\gamma(t_1), \ldots, \gamma(t_d)\}$$

then face numbers bounded by those of $C_{d,n}$

(extremal case) (messy but worked out beautiful in eyes of beholder!)

proved P. McMullen for boundaries of polytopes

R. Stanley for abstract simplicial spheres

famous proof, tied together commutative algebra w/ combinatorics,

"taught combinatorists (and others) the phrase"

Cohen-Macaulay

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in algebraic geometry, many interps

Simplest (?) $I \subset S = k[x_0, \ldots, x_n]$ defines $X \subset \mathbb{P}^n$

S/I Cohen-Macaulay $\iff$ chain of syzygies no longer (same length) as a complete intersection same codim as X

$\iff$ $X \cap L$ pure dim (no embedded components) for generic linear spaces $L \subset \mathbb{P}^n$

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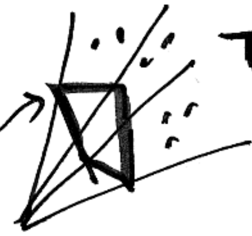
moving toward combinatorics, cell complexes:

take polyhedral cone,  T = semigroup of lattice points

(holes in lattice $\Rightarrow$ not Cohen-Macaulay)

Γ

(in fact Gorenstein)

$K[T] = S/I$
semigroup algebra   toric ring

I defines X ~~toric variety~~ Cohen-Macaulay toric variety (needs choice of gens of T)

Γ = boundary of slice of cone $\cong S^i$ some i

$\Rightarrow H^*(\Gamma) = \begin{cases} K, & * = i \\ 0 & \text{else} \end{cases}$ } computes sheaf cohomology of X,
0 except dim X

So seeing (co)homology ranks of 0, except in top dim

$\Longrightarrow$ Pavlovian response, "there must be something <u>Cohen-Macaulay</u> going on here"

response is usually justified, and rewarded.

<u>Def</u> finite poset P <u>Cohen-Macaulay</u> (over $\mathbb{Q}$)

if for all $x < y$ in $\hat{P}$,

look at $\Delta((x,y))$, (order complex of <u>open interval</u>, he finally adopts my convention)

$$\tilde{H}_i(\Delta((x,y)),\mathbb{Q}) = 0, \quad i < \underbrace{\dim \Delta((x,y))}_{\ell(x,y)-2}$$

$x < z < y$
dim 0, $\ell(x,y) = 2$

def $\boxed{\mu_P \text{ alternates in sign} \iff (-1)^{\ell(x,y)}\mu_P(x,y) \geq 0}$

P CM $\Rightarrow$ μ_P alternates in sign

$\cancel{\mu_{\hat{P}}(x,y) = \tilde{\chi}(\Delta((x,y))) = (-1)^{\dim}\tilde{h}_d(\Delta(x,y),\mathbb{Q})}$

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examples of CM posets $\begin{cases} P(\Gamma),\ \text{"nice" manifolds w or w/o boundary} \\ \text{finite } \underbrace{\text{semimodular}}_{\text{we never looked at these yet.}} \text{ lattices} \end{cases}$

3.9 Lattices and their Möbius Algebras

special methods for lattices

Möbius algebra of L lattice / k field $\quad A(L,k)$

$= \underbrace{k[L]}_{\text{semigroup algebra}}$ view meet operation as sum for L

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familiar example $\mathbb{N}^n$ lattice points in first orthant

separate $\begin{cases} (\mathbb{N}^n,+) & \text{usual semigroup} \\ (\mathbb{N}^n, \underset{\substack{\text{meet} \\ \cong \text{ intersection}}}{\wedge}) = (\mathbb{N}^n, \gcd) & \text{Möbius algebra} \end{cases}$

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So $A(\mathbb{N}^n, k)$ looks like polynomial ring, except multiplication is gcd operation. (!)

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not a quotient of a finite dim poly ring

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idempotent basis $x \in L$ for commutative $A(L,k)$

want to make $\underbrace{A(L,k) \cong k^{\#L}}_{\text{wait}}$ more explicit: (L finite)

$$\boxed{\text{for } x \in L \text{ define } \delta_x \in A(L,k) \text{ by } \delta_x = \sum_{y \le x} \mu(y,x)\, y}$$

$$\Rightarrow \boxed{x = \sum_{y \le x} \delta_y} \quad \text{(very cool) } (**)$$

(**) span, right number $\Rightarrow$ δ_x's form basis for $A(L,k)$

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check (**):

$$\sum_{y \le x} \delta_y = \sum_{y \le x} \sum_{z \le y} \mu(z,y)\, z$$

$$= \sum_{z \le x} \underbrace{\Big(\sum_{z \le y \le x} \mu(z,y) \Big)}_{\substack{0,\ z<x \\ 1,\ z=x}} z = x$$

Theorem L finite lattice,

$A'(L,k) = \coprod_{x \in L} K_x$, $K_x \cong k$

δ'_x identity elem of K_x $\quad \delta'_x \delta'_y = \begin{cases} 0, & x \ne y \\ \delta'_x, & x = y \end{cases}$

then $\Theta : A(L,k) \to A'(L,k)$ algebra isomorphism proof

look at map
$$\begin{array}{ccc} x & \overset{\theta}{\longmapsto} & x' \\ \| & & \| \\ \sum_{y \le x} \delta_y & & \sum_{y\le x} \delta'_y \end{array}$$

need only show $x'y' = (x \wedge y)'$

$$x'y' = \Big(\sum_{z \le x} \delta'_z\Big)\Big(\sum_{w \le y} \delta'_w\Big) = \sum_{\substack{z \le x \\ w \le y \\ z = w}} \delta'_z = \sum_{z \le x \wedge y} \delta'_z = (x \wedge y)'$$

Corollary L finite lattice, $\hat{1} \neq a$

$$\boxed{\sum_{x \mid x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0} \qquad (\dagger\dagger)$$

$a = \hat{0}$ is usual $\sum_x \mu(x, \hat{1}) = \hat{0}$, not new. $a \neq \hat{0}$ new, fewer terms

---

proof in $A(L, \mathbb{C})$,

$$\begin{cases} a\delta_{\hat{1}} = \Big(\sum_{b \le a} \delta_b\Big)\delta_{\hat{1}} = 0 \quad \text{since } a \neq 1 \\ a\delta_{\hat{1}} = a \sum_{x \in L} \mu(x, \hat{1}) x = \sum_{x \in L} \mu(x, \hat{1})\, a \wedge x \end{cases}$$

the $\hat{0}$ coefficient (in expansion $\sum_{x \in L} c_x x$)

is $(\dagger\dagger)$ as above, claimed.