TNB ① 21 Feb 02
comb

[start w/ pages 8-10, Tues 19 Feb 02]

Note Stanley's convention

P poset, $\hat{P}$ has $\hat{0}, \hat{1}$ adjoined (needed or not!)

so $\tilde{\chi}_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$

this is same as us, because $(\hat{0}, \hat{1})$ is P

<u>geometric realization</u> $X = |\Delta|$ of Δ

$$\tilde{\chi}(X) = \sum_i (-1)^i \tilde{h}_i(X, \mathbb{Z})$$

(compute any way you like singular homology of X
simplicial " " Δ)

so $\mu_{\hat{P}}(\hat{0}, \hat{1})$ depends only on $|X|$ not X

("homotopy equivalence of posets" sneaking in...)

<u>finite</u> <u>regular</u> <u>cell</u> <u>complex</u> Γ = union nonempty pairwise disjoint

open cells $\sigma_i \subset \mathbb{R}^N$ so

(a) $(\bar{\sigma}_i, \bar{\sigma}_i - \sigma_i) \cong (\mathbb{B}^n, \mathbb{S}^{n-1})$ some $n = n(i)$

(b) each $\bar{\sigma}_i - \sigma_i$ is union of σ_j's

[<u>Cellular homology</u> generalizes <u>simplicial homology</u>

need sign for $\sigma_j \subset \sigma_i$ facet, sign not obvious

any two "good" sign rules equivalent

(see Bruns, Herzog e.g.)]

$|\Gamma|$ realization of Γ

$sd(\Gamma)$ simplicial complex of chains of cells $\sigma_{i_1} \subset \sigma_{i_2} \subset .. \sigma_{i_\ell}$ of Γ

first barycentric subdivision

$P(\Gamma)$ facet poset of cells of Γ

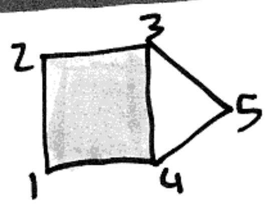
(don't include empty face $\emptyset$)

then $\boxed{sd(\Gamma) = \Delta(P(L))}$

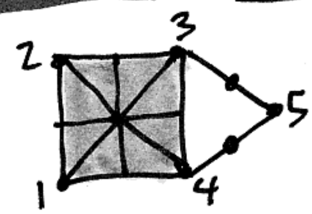
so $\boxed{\mu_{\hat{P}}(\hat{0},\hat{1}) = \tilde{\chi}(|\Gamma|)}$

$-$ empty $= -1$
$+ \ v = 5$
$- \ e = -6$
$+ \ F = 1$
$\boxed{\tilde{\chi} = -1}$

example



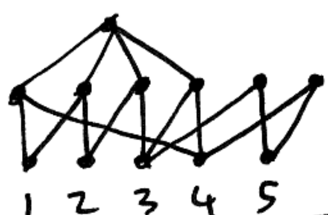
Γ



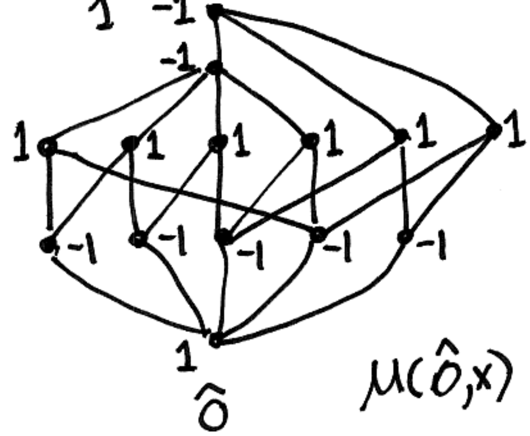
$\underbrace{sd(\Gamma) = \Delta(P(\Gamma))}$

$\left(\begin{array}{l} h_*(S^1) = 0 \ \ * \neq 1 \\ h_1(S^1) = 1 \end{array} \right)$

$P(\Gamma)$



4 maximal chains length 1
8 " " " 2



$\hat{P}$:

recursively compute $\mu_{\hat{P}}(\hat{0},\hat{1})$

$$\mu_{xy} = -\sum_{x \leq z < y} \mu_{xz}$$

$\Rightarrow \boxed{\mu(\hat{0},\hat{1}) = -1}$

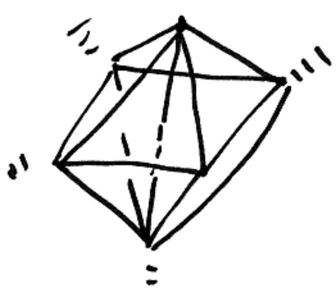
$\mu(\hat{0},x)$ for $x \in \hat{P}$

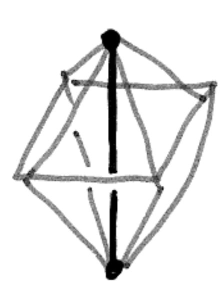
what about other $\mu_{\hat{P}}(x,y)$?

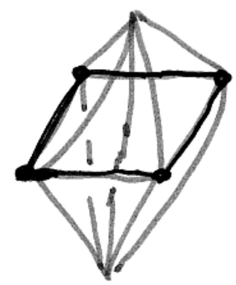
<u>link</u> $\operatorname{lk} F$ of a face $F \in \Delta$, Δ simplicial complex, defined by

$$\operatorname{lk} F = \{ G \setminus F \mid F \subseteq G \in \Delta \}$$ (restated slightly from Stanley)

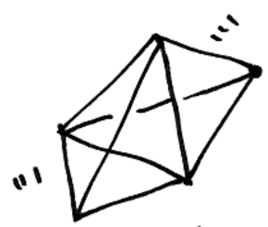
in closed <u>manifold</u>, $\left.\begin{array}{l} \dim F = i \\ \dim \Delta = n \end{array}\right\}$ $\cancel{\dim}$ $\operatorname{lk} F \cong \mathbb{S}^{n-i-1}$


part of Δ, dim 3


F, dim 1


$\operatorname{lk} F \cong \mathbb{S}^1$ (1-sphere)

simplicial manifold, PL category, becomes our "definition" of a manifold
(WARNING: higher dims, wild triangulations of manifolds this fails)
<u>but true at level of homology</u>


part of Δ, dim 3


F, dim 2


$\operatorname{lk} F \cong \mathbb{S}^0$ (2 points)

Γ finite regular cell complex, $|\Gamma| =$ manifold

$$\mu_{\hat{P}}(x,y) = \begin{cases} 0, & x \neq \hat{0},\ y = \hat{1}, \quad x \text{ lies on boundary of } |\Gamma| \ ☑ \\ \tilde{\chi}(\Gamma), & x = \hat{0},\ y = \hat{1} \quad ☑ \\ (-1)^{\ell(x,y)} & \text{otherwise} \end{cases}$$

we use $\mathrm{lk}\,F$ to express $\mu_{\hat{P}}(x,y)$:

given $x<y$ choose saturated chains in P (not $\hat{P}$)

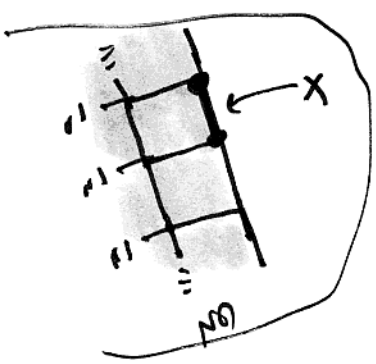
$$\boxed{\underbrace{x_1}_{\min} < x_2 < \dots < x_r = x} < \boxed{y = y_1 < \dots < \underbrace{y_s}_{\max}}$$

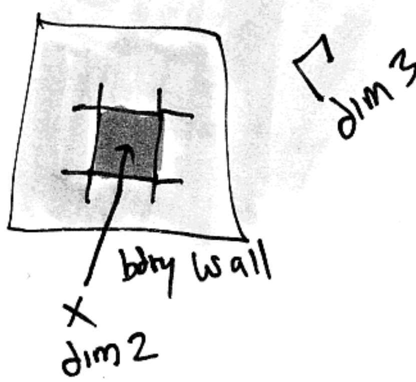
then $\underbrace{\Delta((x,y))}_{\text{our notation so far}} = \mathrm{lk}\,F$ in $\Delta(P)$

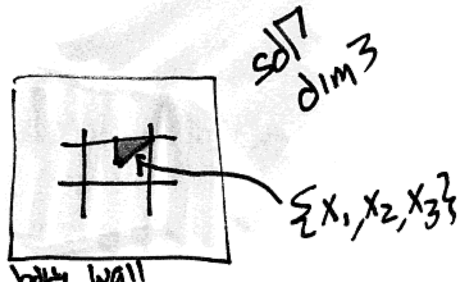
$F = \{x_1, \dots, x_r, y_1, \dots, y_s\}$

(so the family of spaces $\Delta((x,y))$ all related, as links of $\Delta(P)$)

Now, x lies on boundary of Γ







choose

$x_1 < x_2 < x_3 = x$

still boundary cell

$\Rightarrow \mathrm{lk}\,F \cong \mathbb{B}^i$ some i

$$\Rightarrow \mu_{\hat{P}}(x,\hat{1}) = \tilde{\chi}(\Delta((x,\hat{1}))) = \tilde{\chi}(\mathrm{lk}\,F) = 0$$

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remaining case: $\mu_{\hat{P}}(x,y) = (-1)^{\ell(x,y)}$

$x \neq \hat{0} \quad y \neq \hat{1}$ or x not on boundary

How to think of this: $\hat{1}$ has "boundary" Γ

any other y has boundary $\cong S^i$ some i

lk F is usual link of x in boundary of y

so $\tilde{\chi}(\text{lk } F)$ is ± 1, depends on dim of link

homology of a sphere

$\Longrightarrow$ Def P (with $\hat{0}, \hat{1}$) semi-Eulerian

if $\mu_P(x,y) = (-1)^{\ell(x,y)}$ for $(x,y) \neq (\hat{0},\hat{1})$

Eulerian

if also $\mu_P(\hat{0},\hat{1}) = (-1)^{\ell(\hat{0},\hat{1})}$

$|\Gamma|$ manifold w/o boundary $\Longrightarrow$ $\hat{P}(\Gamma)$ semi-Eulerian

$|\Gamma|$ sphere $\Longrightarrow$ " Eulerian

Example B_n Eulerian $\left(B_n \cong 2^n \quad S, T \subset n \quad \mu(T,S) = (-1)^{|S-T|} \right)$

recall

indeed $B_n = \hat{P}(\Gamma)$, Γ boundary complex of $\underbrace{(n-1)}_{\text{dim}}$-simplex

(subsets of $n \Longleftrightarrow$ faces of simplex) aside: <u>upper bound conjecture for spheres</u>

Given any simplicial sphere, d vertices S^{n-1},
pick d generic points on rational normal curve

$$\gamma(t) = (t, t^2, \ldots, t^n) \hookrightarrow \mathbb{R}^n$$

and take $C_{d,n}$ = boundary of convex hull of

$$\{\gamma(t_1), \ldots, \gamma(t_d)\}$$

then face numbers bounded by those of $C_{d,n}$
(extremal case) (messy but worked out / beautiful in eyes of beholder!)

proved P. McMullen for boundaries of polytopes
<u>R. Stanley</u> for abstract simplicial spheres

famous proof, tied together commutative algebra w/ combinatorics,
"taught combinatorists (and others) the phrase"
<u>Cohen-Macaulay</u>

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in algebraic geometry, many interps
<u>simplest</u> (?) $I \subset S = k[x_0, \ldots, x_n]$ defines $X \subset \mathbb{P}^n$

S/I Cohen-Macaulay $\iff$ chain of syzygies no longer (same length) as a <u>complete intersection</u> same codim as X

$\iff$ $X \cap L$ pure dim (no embedded components)
for generic linear spaces $L \subset \mathbb{P}^n$

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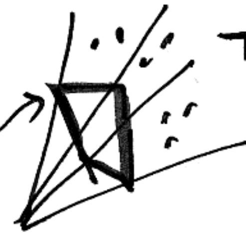
moving toward combinatorics, cell complexes:

take polyhedral cone,  T = semigroup of lattice points

(holes in lattice $\Rightarrow$ not Cohen-Macaulay)

Π

(in fact Gorenstein)

$$\underset{\text{semigroup algebra}}{K[T]} = \underset{\text{toric ring}}{S/I}$$

I defines X toric variety (Cohen-Macaulay) (needs choice of gens of T)

Π = boundary of slice of cone $\cong S^i$ some i

$\Rightarrow H^*(\Pi) = \begin{cases} K, & * = i \\ 0 & \text{else} \end{cases}$ computes sheaf cohomology of X, 0 except $\dim X$

So seeing (co)homology ranks of 0, except in top dim

$\Rightarrow$ Pavlovian response, "there must be something Cohen-Macaulay going on here"

response is usually justified, and rewarded.

Def finite poset P Cohen-Macaulay (over $\mathbb{Q}$)

if for all $x < y$ in $\hat{P}$,

look at $\Delta((x,y))$, (order complex of open interval, he finally adopts my convention)

$$\tilde{H}_i(\Delta((x,y)),\mathbb{Q}) = 0, \quad i < \underbrace{\dim \Delta((x,y))}_{\ell(x,y)-2}$$

$x < z < y$
$\dim 0,\ \ell(x,y) = 2$

def $\boxed{\mu_P \text{ alternates in sign} \iff (-1)^{\ell(x,y)} \mu_P(x,y) \geq 0}$

P CM $\Rightarrow$ μ_P alternates in sign

$\cancel{\mu_{\hat{P}}(x,y) = \tilde{\chi}(\Delta((x,y))) = (-1)^{\dim} \tilde{h}_j(\Delta(x,y), \mathbb{Q})}$

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examples of CM posets $\begin{cases} P(\Gamma), \text{ "nice" manifolds with or w/o boundary} \\ \text{finite } \underbrace{\text{semimodular}}_{\text{we never looked at these yet.}} \text{ lattices} \end{cases}$

3.9 $\boxed{\text{Lattices and their Möbius Algebras}}$

special methods for lattices

<u>Möbius algebra</u> of L lattice / k field $\quad A(L,k)$

$= \underbrace{k[L]}_{\text{semigroup algebra}}$ $\quad$ view <u>meet</u> operation as sum for L

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familiar example $\mathbb{N}^n$ lattice points in first orthant

separate $\begin{cases} (\mathbb{N}^n, +) \quad \text{usual semigroup} \\ (\mathbb{N}^n, \underset{\substack{\text{meet} \\ \cong \text{ intersection}}}{\wedge}) = (\mathbb{N}^n, \gcd) \quad \text{Möbius algebra} \end{cases}$

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So $A(\mathbb{N}^n, k)$ looks like polynomial ring, except multiplication is gcd operation. (!)

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not a quotient of a finite dim poly ring

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idempotent basis $x \in L$ for commutative $A(L,k)$

want to make $\underbrace{A(L,k) \cong k^{\#L}}_{\text{wait}}$ more explicit: (L finite)

$$\boxed{\text{for } x \in L \text{ define } \delta_x \in A(L,k) \text{ by } \quad \delta_x = \sum_{y \le x} \mu(y,x)\, y}$$

$$\Rightarrow \quad \boxed{x = \sum_{y \le x} \delta_y} \qquad \text{(very cool)} \ (**)$$

(**) span, right number $\Rightarrow$ δ_x's form basis for $A(L,k)$

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check (**):

$$\sum_{y \le x} \delta_y = \sum_{y \le x} \sum_{z \le y} \mu(z,y)\, z$$

$$= \sum_{z \le x} \underbrace{\Big(\sum_{z \le y \le x} \mu(z,y) \Big)}_{\substack{0, \ z<x \\ 1, \ z=x}} z = x$$

Theorem L finite lattice,

$A'(L,k) = \coprod_{x \in L} K_x, \quad K_x \cong k$

δ'_x identity elem of K_x $\qquad \delta'_x \delta'_y = \begin{cases} 0, & x \ne y \\ \delta'_x, & x = y \end{cases}$

Then $\Theta : A(L,k) \to A'(L,k)$ algebra isomorphism proof look at map

$$\begin{array}{ccc} x & \overset{\theta}{\longmapsto} & x' \\ \| & & \| \\ \sum_{y\le x} \delta_y & & \sum_{y \le x} \delta'_y \end{array}$$

need only show $x'y' = (x\wedge y)'$

$$x'y' = \Big(\sum_{z\le x} \delta'_z\Big)\Big(\sum_{w\le y} \delta'_w\Big) = \sum_{\substack{z\le x\\ w\le y\\ z=w}} \delta'_z = \sum_{z\le x\wedge y} \delta'_z = (x\wedge y)'$$

~~Theorem~~ Corollary L finite lattice, $\hat{1}\neq a$

$$\boxed{\sum_{x \mid x\wedge a=\hat{0}} \mu(x,\hat{1}) = 0} \quad (\dagger\dagger)$$

$a=\hat{0}$ is usual $\sum_x \mu(x,\hat{1}) = \hat{0}$, not new. $a\neq\hat{0}$ new, fewer terms

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proof in $A(L,\mathbb{C})$,

$$\begin{cases} a\delta_{\hat{1}} = \Big(\sum_{b\le a} \delta_b\Big)\delta_{\hat{1}} = 0 & \text{since } a\neq 1 \\ a\delta_{\hat{1}} = a\sum_{x\in L}\mu(x,\hat{1})x = \sum_{x\in L}\mu(x,\hat{1})\, a\wedge x \end{cases}$$

the $\hat{0}$ coefficient (in expansion $\sum_{x\in L} c_x x$)

is $(\dagger\dagger)$ as above, claimed.