Tues ① 19 Feb 02
comb

[start w/ pages 10–12, Thurs 14 Feb 02]

3.7 Möbius inversion formula

$$\zeta = \sum_{x \le y} [x,y] \quad \text{zeta function}$$

$$\delta = \sum_{x} [x,x] \quad \text{identity (delta function)}$$

$$\mu = \sum_{x \le y} \mu(x,y)\,[x,y] \quad \text{Möbius function}$$

defined by $\boxed{\mu\zeta = \delta}$ inverse to zeta function

$$\underbrace{\left(\sum_{x \le y} \mu_{xy}[x,y]\right)\left(\sum_{x \le y} [x,y]\right)} = \sum_{x} [x,x]$$

$$\sum_{x \le y} \left(\sum_{x \le z \le y} \mu_{xy} \underbrace{[x,z][z,y]}_{[x,y]} \right) = \sum_{x \le y} \delta(x,y)\,[x,y]$$

or

$$\boxed{\mu_{xx} = 1}$$

$$\sum_{x \le z \le y} \mu_{xz} = 0 \implies \boxed{\mu_{xy} = -\sum_{x \le z < y} \mu_{xz} \quad \text{for all } x < y \text{ in } P}$$

Prop (Möbius inversion formula)

P poset so every principal order ideal is finite

$f, g : P \to \mathbb{C}$. Then

$$\boxed{g(x) = \sum_{y \le x} f(y) \quad \forall x \in P}$$

$$\iff \boxed{f(x) = \sum_{y \le x} g(y)\mu(y,x) \quad \forall x \in P}$$

proof: Let $I(P)$ act on $\mathbb{C}^P = \{\text{fns } P \to \mathbb{C}\}$ (on right) as algebra of linear transformations by

$$(f\gamma)(x) = \sum_{y \le x} f(y)\,\gamma(y,x) \qquad \begin{array}{l} f \in \mathbb{C}^P \\ \gamma \in I(P) \end{array}$$

Then

$$\boxed{f\zeta = g} \iff \boxed{f = g\mu}$$

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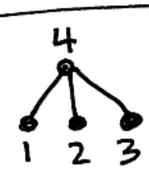
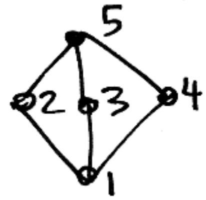
ex: $P =$ 

$$\zeta = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & & & 1 \\ 2 & & 1 & & 1 \\ 3 & & & 1 & 1 \\ 4 & & & & 1 \end{array}$$

$$\mu = \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}$$

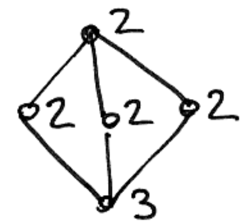
(boring)

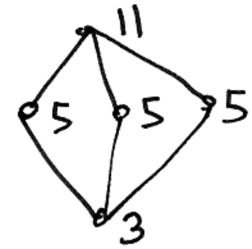
example:

$P =$ 

$$\zeta = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix}$$

$f : P \to \mathbb{C}$: 

$g : P \to \mathbb{C}$: 

$$g(x) = \sum_{y \le x} f(y)$$

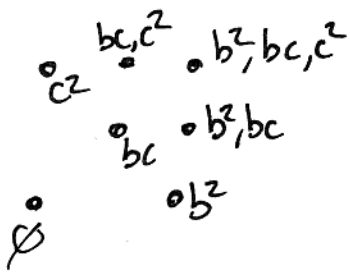
$f\zeta = g$: $$[3\ 2\ 2\ 2\ 2] \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} = [3\ 5\ 5\ 5\ 11]$$

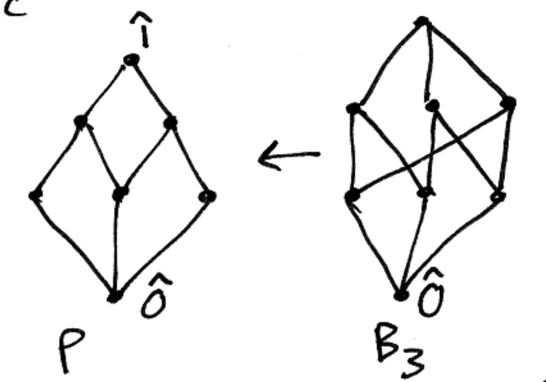
$f = g\mu$: $$[3\ 2\ 2\ 2\ 2] = [3\ 5\ 5\ 5\ 11] \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix}$$

Think of P as set of property states, $\le$ is "fewer or same properties"
(properties not entirely independent)

g counts objects, same or fewer properties, f counts same only return to properties monomials deg d in $k[a,b,c,d]$

divisible by b^2, bc, c^2





$\underbrace{\hspace{4em}}$ special case, all combinations occur independently.

(dual form in book)

3.8 Techniques for computing Möbius functions

ex: $P =$ chain $\mathbb{N} = \{0, 1, 2, \dots\}$

$$\mu_{ij} = -\sum_{i \le k < j} \mu_{ik} \Rightarrow \quad \mu_{ii} = 1, \quad \mu_{i,i+1} = -1, \quad \mu_{i,j} = 0,\ j > i+1$$

(we held off on computing μ_{B_n})

$$\boxed{g(n) = \sum_{i=0}^{n} f(i) \ \forall n \iff f(n) = g(n) - g(n-1) \ \forall n}$$

finite difference calculus Σ, Δ

$$\boxed{g = \Sigma f \iff f = \Delta g}$$

with Σ, Δ suitably defined

Prop (product formula) P, Q $\underbrace{\text{locally finite posets}}_{\text{always our min hypothesis}}$

$P \times Q$ direct product

if $(x,y) \le (x',y')$ in $P \times Q$ then

$$\boxed{\mu_{P\times Q}((x,y),(x',y')) = \mu_P(x,x')\,\mu_Q(y,y')}$$

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proof
$$\sum_{(x,y)\le(u,v)\le(x',y')} \mu_P(x,u)\mu_Q(y,v) = \Big(\sum_{x\le u\le x'} \mu_P(x,u)\Big)\Big(\sum_{y\le v\le y'} \mu_Q(y,v)\Big)$$
$$= \delta_{xx'}\,\delta_{yy'} = \delta_{(x,y),(x',y')}$$

but we know that $\boxed{\displaystyle\sum_{x\le z\le y} \mu_{x,z} = \delta_{xy}}$

forcing the formula to hold, inductively.

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example B_n boolean lattice rank n $\#B_n = 2^n$

looks like n-cube

B_3

$B_n \cong 2^n$ μ for chain $2 = \{1,2\}$:

$$\zeta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \mu = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

identify B_n with all subsets of $n = \{1,\dots,n\}$

$\Rightarrow$ [redacted] $\mu(T,S) = (-1)^{|S-T|}$

product of 1's S and T agree

-1's S and T differ

but $T \subseteq S$

so for $B_n \cong 2^n$, specializes to

$$g(S) = \sum_{T \subseteq S} f(T)$$

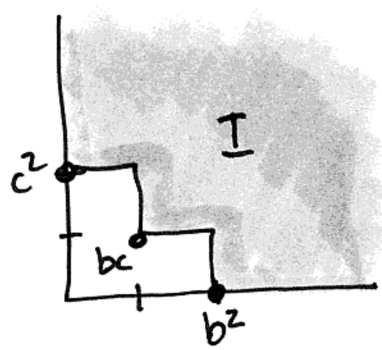
$$\Leftrightarrow \quad f(S) = \sum_{T \subseteq S} (-1)^{|S-T|} g(T)$$

f counts exactly props
g counts exactly or fewer
(dual: or more)

<u>example</u>: $\mathbb{N}^n = \underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{n \text{ copies}}$

$$\mu(x,y) = \begin{cases} (-1)^{\sum y_i - x_i}, & y-x \text{ 0-1 vector} \\ 0, & \text{else} \end{cases}$$

example: $I = (b^2, bc, c^2) \subseteq S = k[b,c]$



let $g_I : \mathbb{N}^2 \to \mathbb{C}$ be characteristic function of I

~~h(z) = { 0, z exponent of I~~

$$g_I(z) = \begin{cases} 1, & z \text{ exponent of } I\text{: } b^{z_1}c^{z_2} \in I \\ 0, & \text{else} \end{cases}$$

~~for a principal ideal (b^2):~~

~~$g_{(b^2)}(z) = \begin{cases} 1, & z \geq (2,0) \\ 0, & \text{else} \end{cases}$ not useful~~

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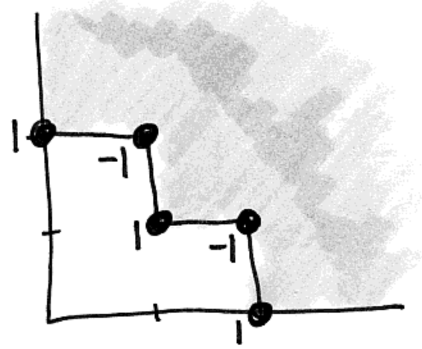
Now ~~looks like~~ want to express

$$\boxed{g_I(z) = \sum_{y \le z} f_I(y)}$$ (as before)

this is $g = f\zeta \Leftrightarrow f = g\mu$ so

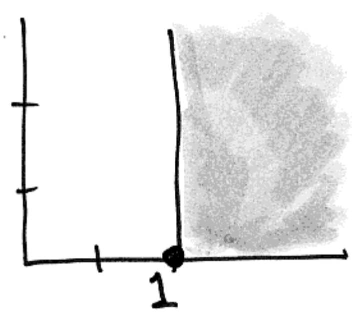
$$\boxed{f_I(z) = \sum_{y \le z} g_I(y)\,\mu(y,z)}$$



$f_I \neq 0$ at points •

How does this relate to Hilbert functions?
(we've seen one interp. This is another)

for a principal ideal (b^2) obvious that $f_{(b^2)}(z) = \begin{cases} 1, & z=(2,0) \\ 0, & \text{else} \end{cases}$



$f_{(b^2)}$

but $\displaystyle\sum_{z \in \mathbb{N}^2} g_{(b^2)}(z)\, b^{z_1} c^{z_2} = \frac{b^2}{(1-b)(1-c)}$

so associating $\dfrac{b^2}{(1-b)(1-c)}$ to b^2

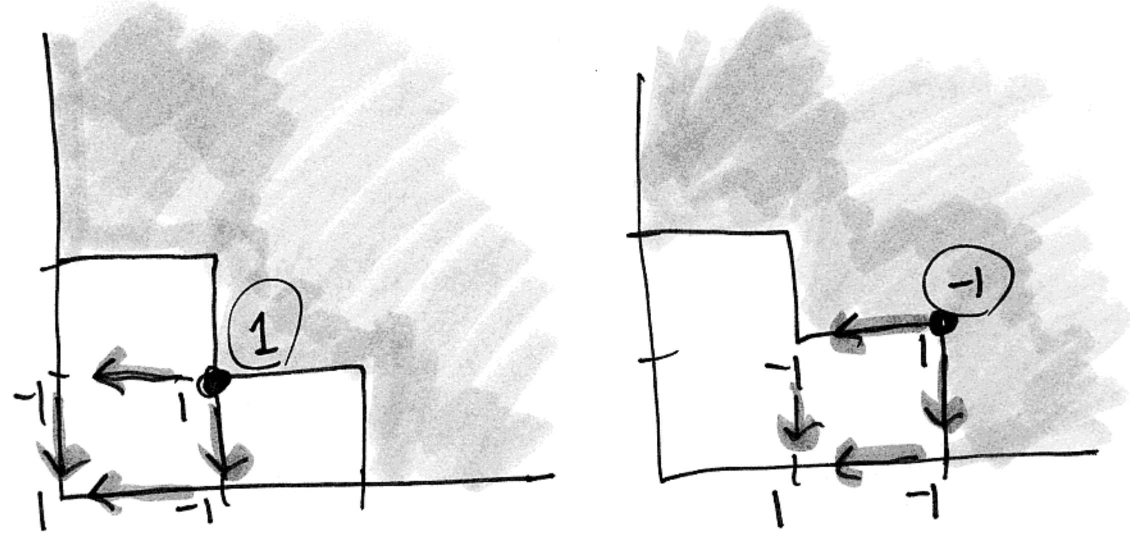
is like computing g from f

$$\boxed{\sum_{z \in \mathbb{N}^2} \dim(I_z)\, b^{z_1} c^{z_2} = \frac{b^2 + bc + c^2 - b^2c - bc^2}{(1-b)(1-c)}}$$

is just adding up, using f_I, to get g_I
in generating function form

This is helpful, because computing f_I is local question

$$\mu(x,y) = \begin{cases} (-1)^{\sum y_i - x_i}, & y-x \text{ 01 vector} \\ 0, & \text{else} \end{cases}$$



$f_I = g_I \mu$

define $K_z = \{ F \subset \{1..n\} \mid z-F \text{ exponent of } I \}$

for $I \subseteq k[x_1, \ldots, x_n]$

K_z is simplicial complex

$-\tilde{\chi}(K_z)$ is just this computation $f_I(z)$

$\Longrightarrow$ $\tilde{H}_i(k, K_z)$ gives individual Betti #'s (ranks of syzygies)

refinement

P = poset of all positive integers, $x \le y \Leftrightarrow x \mid y$
divisibility

$P = J_f(\mathbb{P} + \mathbb{P} + \ldots) = \prod_{n\ge1} \mathbb{N}$ restricted direct product (only finitely many entries $\neq 0$)
finitary

$n = (n_1, n_2, \ldots) \in \prod_{n \ge 1} \mathbb{N}$

$\Leftrightarrow \quad p_1^{n_1} p_2^{n_2} \ldots \in \mathbb{P}$ prime factorization of $\mathbb{P}$
$p_1, p_2, p_3, \ldots = 2, 3, 5, \ldots$ primes

$$\boxed{g(n) = \sum_{d \mid n} f(d) \Leftrightarrow f(n) = \sum_{d \mid n} g(d)\, \mu(n/d)}$$

where $\mu(\frac{n}{d}) = \begin{cases} (-1)^t, & n/d \text{ product } t \text{ distinct primes} \\ 0, & \text{else} \end{cases}$

is classic Möbius ~~[illegible]~~ inversion formula of number thy.

Prop P finite poset w/ $\hat{0}, \hat{1}$

c_i = # chains $\hat{0} = x_0 < x_1 < \ldots < x_i = \hat{1}$
$(c_0 = 0,\ c_1 = 1,\ c_2 = \#P - 2, \ldots)$

$\Rightarrow \quad \mu_P(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \ldots$

proof

$$\mu_P(\hat{0},\hat{1}) = [1+(\zeta-1)]^{-1}(\hat{0},\hat{1})$$

$$= [1-(\zeta-1)+(\zeta-1)^2-\ldots](\hat{0},\hat{1})$$

(note: $1 \in I(P)$!) (recall $(\zeta-1)$ is nilpotent on finite P)

$$= \delta(\hat{0},\hat{1}) - (\zeta-1)(\hat{0},\hat{1}) + (\zeta-1)^2(\hat{0},\hat{1}) - \ldots$$

$$= c_0 - c_1 + c_2 - c_3 + \ldots$$

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topological interpretation

open interval $(x,y) \subset P$ is induced subposet $\{z \in P \mid x<z<y\}$

$\Delta((x,y))$ = order complex of chains in (x,y)

promote to chains in closed interval $[x,y]$

$$\boxed{\begin{array}{c} \{x_1 < \ldots < x_{i-1}\} \in \Delta((x,y)) \\ \Longleftrightarrow \quad x = x_0 < x_1 < \ldots < x_{i-1} < x_i = y \text{ in } [x,y] \end{array}}$$

empty chain $\emptyset \in \Delta((x,y))$

$\Longleftrightarrow \quad x = x_0 < x_1 = y$ in $[x,y]$

accounts for $c_1 = 1$

$\left\langle \begin{array}{l} \tilde{\chi}(\Delta((x,y)) \text{ counts } \emptyset \text{ as } -1 \\ \text{prop computes } \ldots - c_1 + \ldots \end{array} \right\rangle$ reduced Euler characteristic

so $\boxed{\mu(x,y) = \tilde{\chi}(\Delta((x,y)))}$ for $x<y$

$= 1$ for $x=y$