3.5 Chains in distributive lattices

\# k-elem order ideals $I \subseteq P$

$\stackrel{\cdot}{=}$ # elems of $J(P)$ of rank k ← define!

\# k-elem antichains of P

$= \# \{ x \in J(P) \mid x \text{ covers exactly } k \text{ elems} \}$

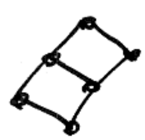
3.5.1 P finite poset, $m \geq 0$

(a) $\# \{ \sigma : P \to m \text{ order preserving} \}$

$=$ (b) $\# \{ \hat{0} = I_0 \leq I_1 \leq \ldots \leq I_m = \hat{1} \}$ in $J(P)$

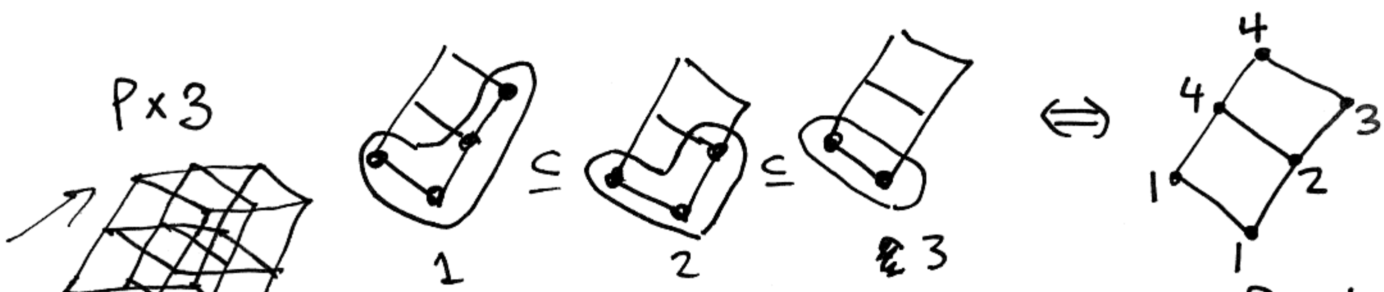
$=$ (c) $\# J(P \times m\text{-}1)$

order ideal in $P \times m\text{-}1$ is a nested chain of order ideals in P:

$P =$ , $m = 4$

$\hat{0} = I_0 \leq I_1 \leq I_2 \leq I_3 \leq I_4 = \hat{1}$

$P \times 3$



I_3 I_2 I_1 $\sigma : P \to 4$

so $I \subset P \times m\text{-}1$ given by $I = \bigcup_{i=1}^{m-1} I_{m-i} \times i$

3.5.2

(a) # surjective order-pres $\sigma : P \to m$

$=$ (b) # chains $\hat{0} = I_0 < I_1 < \dots < I_m = \hat{1}$ in $J(P)$

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if $\#P = n$, $\overset{\text{def}}{=}$ $e(P) = \#\{\sigma : P \to n\}$

order pres bijection

extensions of P to total order

<u>linearizations</u> of P

linear extensions of P

$e(P) =$ # maximal chains in $J(P)$

$=$ # facets in $\underbrace{\Delta(J(P))}_{\text{pure dim simplicial complex}}$

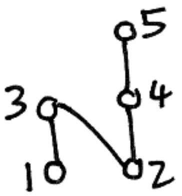
$=$ <u>degree</u> of corresponding variety

of face ring of $\Delta(J(P))$

chain ring of $J(P)$

<u>lattice paths</u>

linear extension is restricted permutation

$P =$ 

$\boxed{\begin{array}{l} \# \pi \in S_5 \text{ so} \\ \pi(1) < \pi(3) \\ \pi(2) < \pi(3) \\ \pi(2) < \pi(4) \\ \pi(4) < \pi(5) \end{array}}$

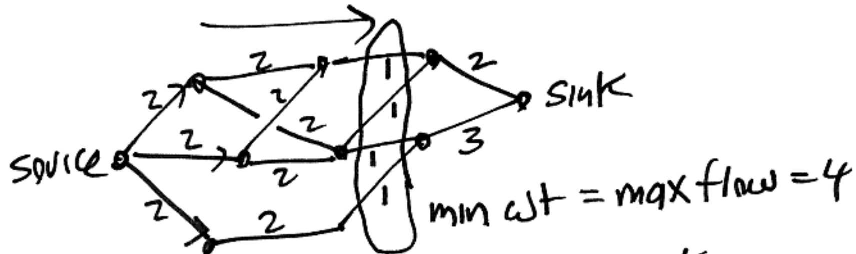
$= e(P)$

Dilworth Theorem

$\min k : P = C_1 \cup \ldots \cup C_k$ partition into chains

$= \max k : k = \#$ antichain of P

reminds us of "min cut / max flow" theorems for network flows



directed graph, source, sink,
each edge has capacity,
max flow source to sink = min cut

oriented matroid theory, linear programming duality

given $P = C_1 \cup \ldots \cup C_k$ (not nec. minimal)

define $\delta : J(P) \longrightarrow \mathbb{N}^k$ by

$\delta(I) = (\# I \cap C_1, \# I \cap C_2, \ldots, \# I \cap C_k)$

$\delta(\hat{0}) = \vec{0}$ $\delta(\hat{1}) = (\# C_1, \ldots, \# C_k)$

$I \subset J$ in $J(P)$ $J = I \cup \{x\}$, $x \in C_i$

J covers I

$\Rightarrow \delta(J) - \delta(I) = (0, \ldots 0, 1, 0, \ldots, 0)$ ↑ ith coord

so δ is cover-preserving $\Rightarrow$ rank-preserving injective lattice homomorphism define $\Gamma_\sigma \subset \mathbb{R}^K$ by

$\left(\begin{array}{l} \text{for each } [I,J] \text{ interval in } J(P) \\ \qquad\qquad \cong \text{ boolean algebra,} \\ \text{take convex hull of image in } \mathbb{N}^K \end{array} \right)$

union is compact polyhedral set,

lattice paths from $\vec{0}$ to $\delta(\hat{1})$ in Γ_σ

$=$ # maximal chains in $J(P)$

$=$ $e(P)$

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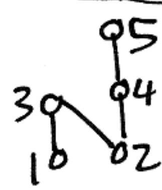
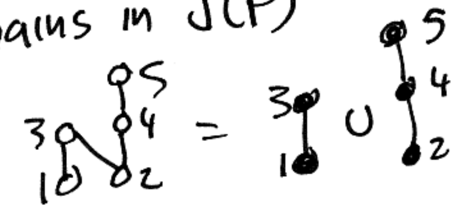
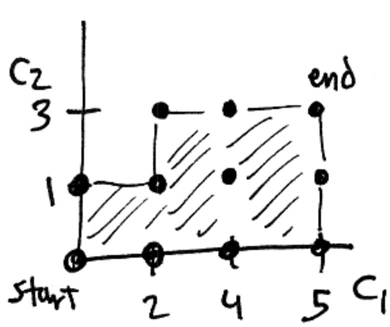
ex: $\#\left\{ \pi \in S_5 \;\middle|\; \begin{array}{l} \pi(1) < \pi(3) \\ \pi(2) < \pi(3) \\ \pi(2) < \pi(4) \\ \pi(4) < \pi(5) \end{array} \right\}$

$= e(P)$, $P =$ 

$=$ # maximal chains in $J(P)$

write $P = C_1 \cup C_2$, 



$J(P) \hookrightarrow \mathbb{N}^2$, and Γ_δ

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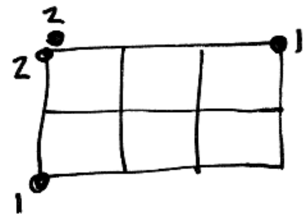
we have this technology

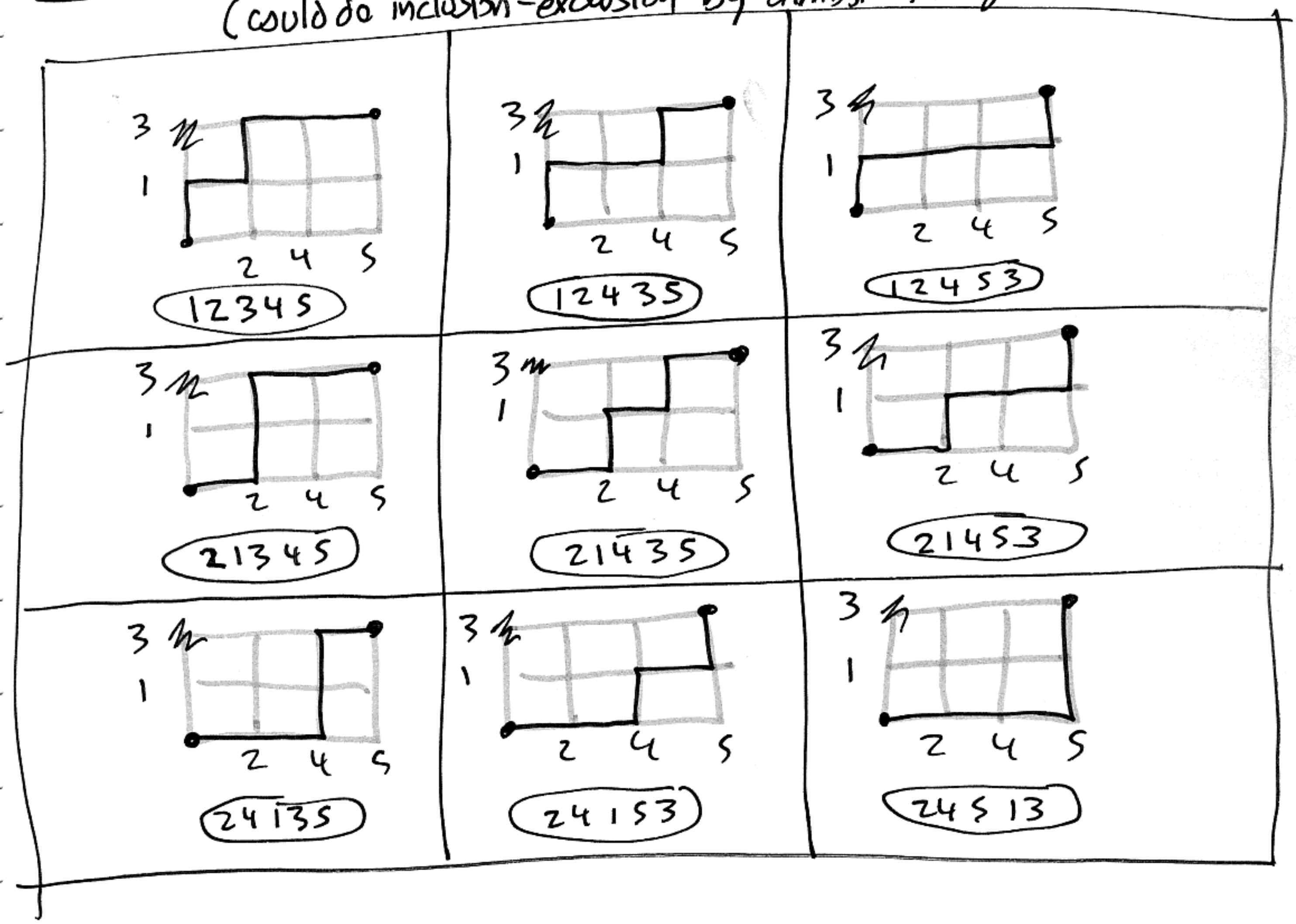


?

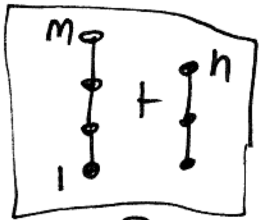
$$\det \begin{vmatrix} \#1\text{to}1 & \#1\text{to}2 \\ \#2\text{to}1 & \#2\text{to}2 \end{vmatrix} = \begin{vmatrix} \binom{5}{3} & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 1 \\ 1 & 1 \end{vmatrix} = 9$$

because permuting ends forces overlaps, whenever points to avoid.

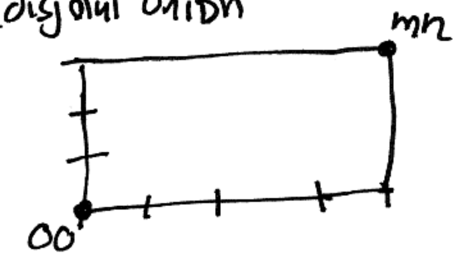
(could do inclusion-exclusion by "almost" triangular matrix



ex: $P = \underset{m}{C_1} + \underset{n}{C_2}$ (disjoint union)



$\Rightarrow$



P $\qquad$ $J(P)$ $\qquad$ $e(P) = \binom{m+n}{m}$

$P = P_1 + \dots + P_k \qquad \# P_i = n_i$

$$\Rightarrow \quad e(P) = \binom{n_1 + \dots + n_k}{n_1, \dots, n_k} e(P_1) \cdots e(P_k)$$

reminiscent of multiset permutations

$$M = \{1^{n_1}, \dots, k^{n_k}\} \qquad n = \sum n_i$$

$$\Rightarrow \quad \phi : \mathcal{S}(M) \times \mathcal{S}_{n_1} \times \dots \times \mathcal{S}_{n_k} \to \mathcal{S}_n$$

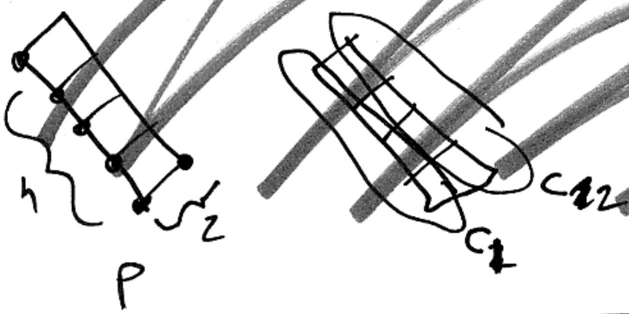
is a bijection

so above formula is restricted version of <u>same</u> construction comb THUR 14 Feb 02

ex: $P = 2 \times n$ $\quad C_1 = \{(1,j)\}$

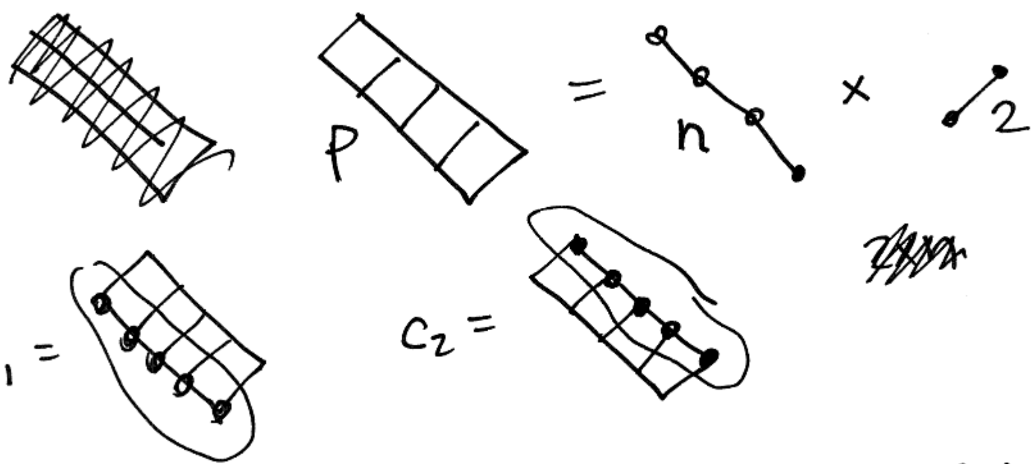
$C_2 = \{(2,j)\}$

why did stanley switch 1,2 here and drawings?



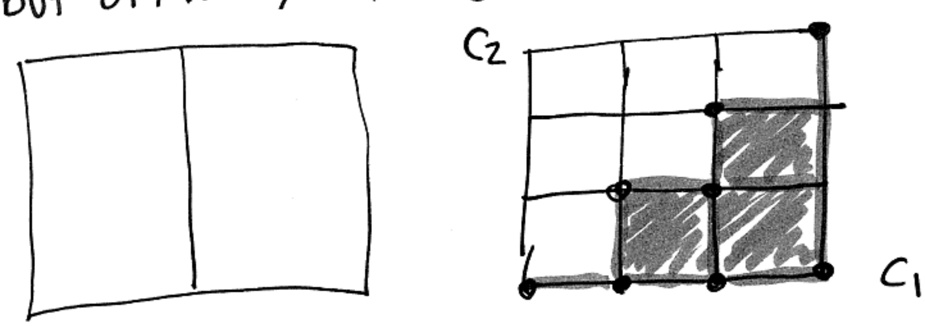
$I \in J(P)$

can't have $\#I \cap C_2 > \#I \cap C_1$



$I \in J(P)$, can't have $\#I \cap C_2 > \#I \cap C_1$

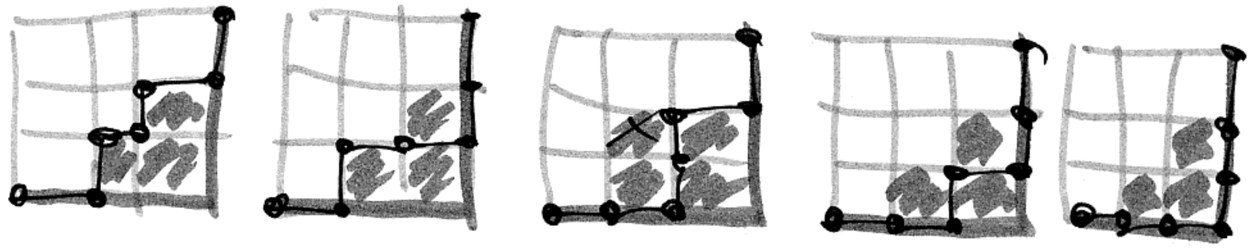
but otherwise, anything goes



$n = 3$

$P = n \times 2$

ex: 3×2



$$\begin{array}{c} c_1 \\ c_2 \end{array}
\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$$

1st row: when in time does c_1 fill in?
2nd row: " " " c_2 " "?

$$e(n\times 2) = \frac{1}{n+1}\binom{2n}{n}$$ famous <u>Catalan numbers</u>

$$e(3\times 2) = \frac{1}{4}\binom{6}{3} = 5 \ \boxed{\checkmark}$$

View e as function on $J(P)$

$e : J(P) \longrightarrow \mathbb{N}$

$e(\hat{1})$ = # extensions of P

$e(I)$ = # extensions of I

then $$e(I) = \sum_{I \text{ covers } I'} e(I')$$

generalizes Pascal's triangle

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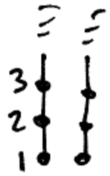
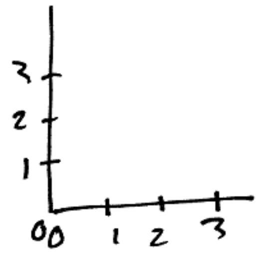
ex: $P = \mathbb{N} + \mathbb{N}$

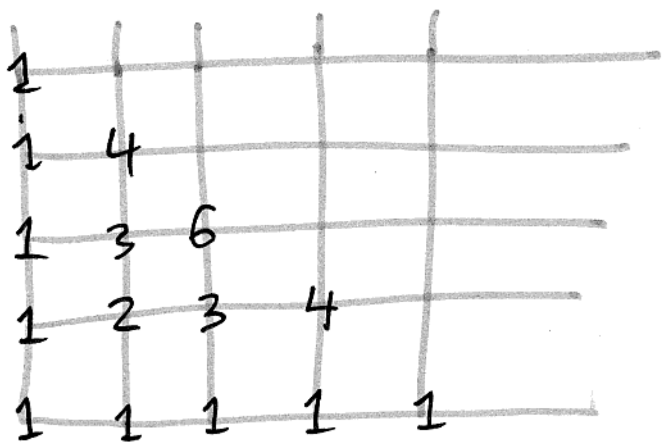


$J(P) = \mathbb{N}^2$

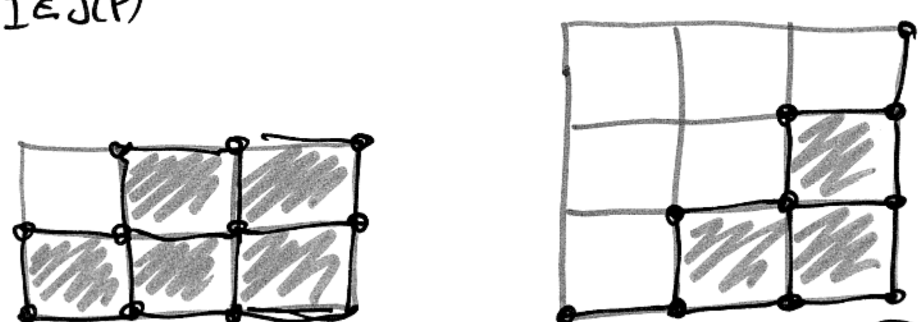


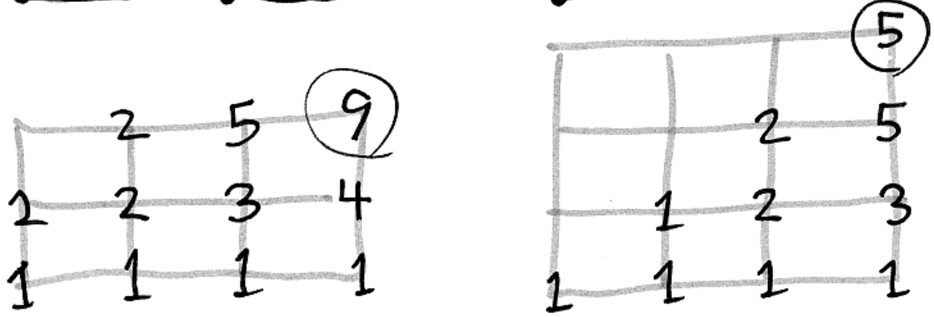
number from 1,
(we're actually taking $P = \mathbb{P} + \mathbb{P}$)

get:



$e(I)$, $I \in J(P)$





generalized Pascal's triangle

~~gene~~ $L = J(P)$, $e : L \to \mathbb{P}$

count π / count lattice paths / satisfy recurrence.

3.6 Incidence Algebra of locally finite poset

(if we get to it)

(locally finite: intervals are finite, e.g. $\mathbb{N}^n$)

$I(P,k) = \{ \text{functions } f : \{\text{intervals of } P\} \to k \}$

$I(P)$ we suppress k, usually take $\mathbb{C}$

$$f = \sum_{x \leq y} f_{xy} [x,y]$$

$\boxed{f(x,y) \text{ for } f_{xy}}$

where $f_{xy} \in k$, $[x,y]$ formal symbol

$$[x,y] \cdot [z,w] = \begin{cases} [x,w], & y = z \\ 0 & y \neq z \end{cases}$$

convolution fg defined by this rule, linearity.

Makes $I(P)$ into a k-algebra.

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if P finite choose linearization $P \to n$

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ & f_{22} & f_{23} & f_{24} \\ & & f_{33} & f_{34} \\ & & & f_{44} \end{bmatrix}$$

with holes, $x_i \not\leq x_j$

$I(P) \cong$ algebra of these matrices

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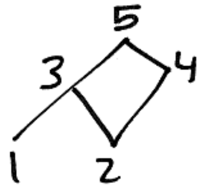

*		*		*
	*	*	*	*
		*		*
			*	*
				*

$\delta = \sum_{x \in P} [x,x]$ identity

Prop 3.6.2 $f \in I(P)$

f left inverse / right inverse / inverse

$\iff f(x,x) \neq 0 \quad \forall x \in P$ see book or (exercise)

ζ zeta function defined by
(fill in $*$'s with 1's)

$\zeta(x,y) = 1$, all $x \leq y$

$\zeta^2(x,y) \left(\sum_{x \leq z \leq y} [x,z][z,y] = \sum_{x \leq z \leq y} [x,y] \right)$

$\quad \| $

$\# [x,y]$

$= \#$ multichains $x = x_0 \leq x_1 \leq x_2 = y$

$\zeta^k(x,y) = \#$ multichains $x = x_0 \leq x_1 \leq \ldots \leq x_k = y$

$= \#$ degree $k-1$ monoms in chain ring of (x,y)

face ring of $\Delta(x,y)$

$$\boxed{\sum_{m=0}^{\infty} \dim(\underbrace{S/I}_{\substack{k[\Delta(P)] \\ = k[P]}})_m t^m = \sum_{m=0}^{\infty} \zeta^{m+1}(x,y) t^m}$$

$$(\zeta-1)(x,y) = \begin{cases} 1, & x<y \\ 0, & x=y \end{cases}$$

$(\zeta-1)^n(\hat{0},\hat{1})$ for $J(P)$
$=$ # linearizations of P
$\#P=n$

$$(\zeta-1)^k(x,y) = \#\text{ chains } x=x_0<x_1<\dots<x_k=y$$

so 3.5 gives additional interps when P is dist lattice

$$(2-\zeta)(x,y) = \begin{cases} 1, & x=y \\ -1, & x<y \end{cases}$$

$2-\zeta$ invertible.

Thm $(2-\zeta)^{-1}(x,y) =$ total # chains (any k) $x=x_0<\dots<x_k=y$

$\ell =$ length of longest chain in $[x,y]$

$\Rightarrow (\zeta-1)^{\ell+1}(u,v)=0 \quad \forall u,v \quad x\le u\le v\le y$

so $(2-\zeta)[1+(\zeta-1)+(\zeta-1)^2+\dots+(\zeta-1)^\ell](u,v)$

$= (1-(\zeta-1))[\quad " \qquad " \qquad " \quad](u,v)$

$= [1-(\zeta-1)^{\ell+1}](u,v)$

$= \delta(u,v)$ identity in $I(P)$

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similar $\quad \eta(x,y) = \begin{cases} 1 & y \text{ covers } x \\ 0 & \text{else} \end{cases}$

$(1-\eta)^{-1}(x,y) =$ total # maximal chains in $[x,y]$