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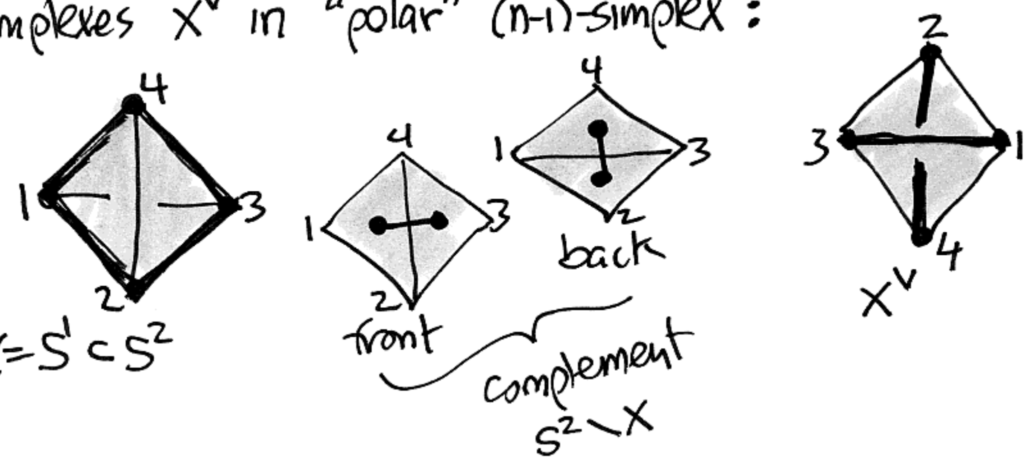
Recall Alexander duality for spheres:

> **Thm** $X \subseteq S^{n-2}$ proper, nonempty subset
>
> $$\Rightarrow H_i(X;G) \cong H^{n-i-3}(S^{n-2} \setminus X;G)$$
> $$H^i(X;G) \cong H_{n-i-3}(S^{n-2} \setminus X;G)$$

Think of S^{n-2} as boundary of standard $(n-1)$-simplex on $\{1..n\}$

Complements of simplicial complexes $X \subset S^{n-2}$ retract to complexes $X^\vee$ in "polar" $(n-1)$-simplex:



This operation can be combinatorially defined:

> **Def** The Alexander dual $X^\vee$ of $X \subseteq 2^{[n]}$
>
> is given by
>
> $$X^\vee = \{F \mid F^c \notin X\} = \{F \mid F \notin X^c\}$$

(complement the sets in $\{1..n\}$ and the collection in $2^{[n]}$, in either order)

(see http://www.math.columbia.edu/~bayer/papers/Duality_B96.pdf)

Then same thm holds without restriction on ~~ser~~ simplicial complexes $X \subseteq 2^{[n]}$:
(we restrict to fields k)

$$\boxed{\begin{array}{ll} \underline{\text{Thm}} & H_i(X;k) \cong H^{n-i-3}(X^\vee;k) \\ & H^i(X;k) \cong H_{n-i-3}(X^\vee;k) \end{array}}$$

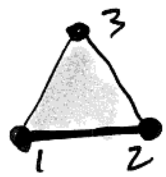
<u>face ring</u> of simplicial complex $X \subset 2^{[n]}$

monomial in $S = k[x_1, \ldots, x_n]$ $\quad x_1^{a_1} \cdots x_n^{a_n}$

$\Longleftrightarrow$ multiset in $[n]$ $\quad \{\underbrace{1, \ldots, 1}_{a_1}, \ldots, \underbrace{n, \ldots, n}_{a_n}\}$

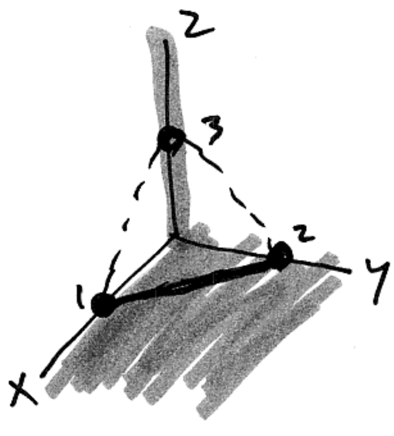
nonzero monomials in $k[X]$
are multisets supported on faces of X.

$\Rightarrow k[X] = S/I$, I generated by $\underbrace{\text{min nonfaces of } X}_{\text{squarefree monomials}}$

ex: $\underline{X} =$ 

$= \{12, 3\}$

min nonfaces are $13, 23 \Longleftrightarrow I = (xz, yz) \subset S = k[x,y,z]$



I cuts out
xy-plane
$\cup$ z-axis
as variety

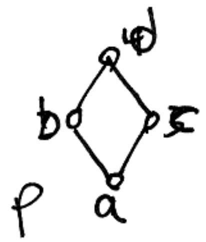
similarly, <u>chain ring</u> of a poset P

nonzero monomials in $k[P]$
are multisets supported on chains of P
$\iff$ multichains in P

$\Rightarrow k[P] = S/I$, $\quad S = k[x_1, \ldots, x_n]$ $\quad x_i \iff$ elems of P

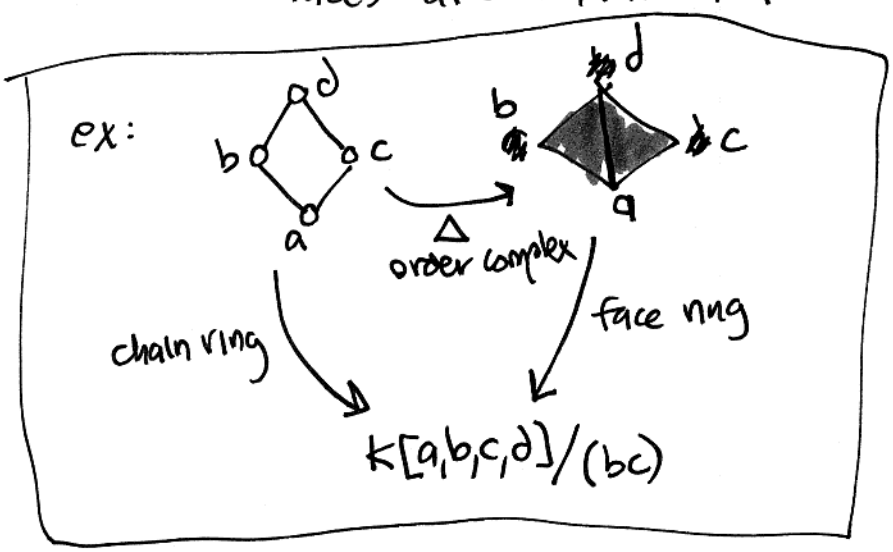
I generated by $\underbrace{\text{incomparable pairs}}_{\text{deg 2 squarefree monoms}}$ in P

ex:



$$k[P] = k[a,b,c,d]/(bc)$$

<u>order complex</u> $\Delta(P)$ of P = simplicial complex
faces are chains in P



what are morphisms in these categories,
do these constructions respect morphisms?

(why did I introduce Alexander duality?)

$\mathcal{A}_{\Rightarrow}$ = category of finite posets
morphism $f: P \to Q$ is order preserving
$$x \le y \Rightarrow f(x) \le f(y)$$

arbitrary maps at set level, order "increases"

$\mathcal{A}_{\Leftarrow}$ = category of finite posets,
morphism $f: P \to Q$ is inverse order preserving
$$x \le y \Leftarrow f(x) \le f(y)$$

$f(x) = f(y) \Rightarrow x \le y$ and $x \ge y \Rightarrow x = y$
maps are injective, order "decreases"

$\mathcal{B}$ = category of finite simplicial complexes
morphism ~~[illegible]~~ $f: X \to Y$
vertices map to vertices
induces map on faces, face must map to a face

$\mathcal{C}$ = category of monomial ideal quotients
morphism $f: R \to T$
ring homomorphism mapping
variables to variables ~~monomials to monomials~~

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So we have functors

$$\boxed{\underset{\text{order-pres}}{\mathcal{A}_{\Rightarrow}} \to \underset{\text{complexes}}{\mathcal{B}}} \quad \text{and} \quad \boxed{\underset{\text{inv. order pres.}}{\mathcal{A}_{\Leftarrow}} \to \underset{\text{rings}}{\mathcal{C}}}$$

$\mathcal{A}_{\Rightarrow} \to \mathcal{B}$: $f : P \to Q$ induces $f_* : \Delta(P) \to \Delta(Q)$

same maps at set level

order preserving $\Rightarrow$ chains map to multichains
$\Rightarrow$ faces map to faces (collapsing allowed)

$\mathcal{A}_{\Leftarrow} \to \mathcal{C}$: $f : P \to Q$ induces $f_* : k[P] \to k[Q]$

set map induces variable map

don't want $0 \mapsto$ nonzero, so

need $f(x_1) \le \dots \le f(x_k)$
$\Rightarrow x_1 \le \dots \le x_k$

---

if we fix $n = \#P$, make all maps identity at set level, then get

$$\begin{tikzcd} & \mathcal{C} & \\ \mathcal{A}_{\Rightarrow} \arrow[ur, "\vee"] \arrow[rr] & & \mathcal{B} \arrow[ul, "\vee"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathcal{C} \arrow[dr, leftrightarrow, "\vee"] & \\ \mathcal{A}_{\Leftarrow} \arrow[ur] \arrow[rr, "\vee"] & & \mathcal{B} \end{tikzcd}$$

commute, where $\vee$ is Alexander dual

(How does this generalize?!?!)

So now examine a few basic defs categorically, make sure they check out...

$P \oplus Q$ <u>ordinal sum</u> treat P, Q as disjoint underlying sets
elems incomparable if $x \in P$, $y \in Q$
inherit order if both in P, both in Q

$$\begin{tikzcd} P \arrow[dr, "i"] \arrow[drr, bend left, "f"] & & \\ & P \oplus Q \arrow[r, "h"] & R \\ Q \arrow[ur, "j"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

(coproduct)

Universal property,
given pair of morphisms
$f : P \to R$, $g : Q \to R$
factors uniquely through $P \oplus Q$

$Q \Rightarrow$ (order-preserving): $\boxed{\checkmark}$ (obvious?)
$Q \Leftarrow$ (inverse order preserving): false

$P = \{x\}$ $Q = \{y\}$ $R = \begin{array}{c} y \\ | \\ x \end{array}$ $x < y$

$P \oplus Q = \begin{array}{cc} \circ & \circ \\ x & y \end{array}$

and $P \oplus Q \xrightarrow{id} R$ is order-preserving
<u>not</u> inverse order preserving

(17) comb class

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$P \times Q$ direct product $\quad \{(x,y) \mid x \in P,\ y \in Q\}$

$(x,y) \le (z,w)$

$\iff \quad x \le z$ in P
$\quad\quad\ \ y \le w$ in Q

(e.g. $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ as posets)

$$\boxed{\begin{tikzcd} P & & \\ & P\times Q \arrow[ul, "i"'] \arrow[dl, "j"] & R \arrow[l, "h"'] \arrow[ull, bend right, "f"'] \arrow[dll, bend left, "g"] \\ Q & & \end{tikzcd}}$$

(product)

Universal property,
given pair of morphisms
$f : R \to P, \quad g : R \to Q,$
factors uniquely through $P \times Q$

h defined by $\quad x \in R \longmapsto (f(x), g(x)) \in P \times Q$

$\mathcal{O}_{\Rightarrow}$ (order preserving) : $\boxed{\checkmark}$

$\mathcal{O}_{\Leftarrow}$ (inverse order preserving) : $\boxed{\text{NO! } i,j \text{ aren't morphisms in this category}}$

$\Rightarrow$: $x \le y$ in $R \Rightarrow \overbrace{f(x) \le f(y),\ g(x) \le g(y)}^{\text{need both}} \Rightarrow h(x) \le h(y)$

$\Leftarrow$: $h(x) \le h(y)$ in $P \times Q \ \cancel{\Rightarrow}\ f(x) \le f(y),\ g(x) \le g(y) \Big\}$ either suffices
$\Rightarrow x \le y$

because maps i, j are special,
not simply because $\underbrace{\text{inverse order preserving}}_{\text{wouldn't suffice}}$

$\boxed{\text{So two categories } \mathcal{O}_{\Rightarrow}, \mathcal{O}_{\Leftarrow} \text{ have different feels...}}$

One could consider $\mathcal{O}_{\Leftrightarrow}$, embeddings $f : P \to Q$ as induced subposet.
$\mathcal{O}_{\Leftarrow}$ comes out a pretty lame category, but is needed for $\mathcal{O}_{\Leftarrow} \to \mathcal{C}$.
What's up here?

BACK TO DISTRIBUTIVE LATTICES:

recall lattice is poset P w/ $\vee, \wedge$

(think $\cup, \cap$ of subsets of a set)

$\leq$ is $\subseteq$

$\vee, \wedge$
- assoc
- commutative
- idempotent $x \vee x = x \wedge x = x$
- absorbtion laws $x \wedge (x \vee y) = x \vee (x \wedge y) = x$
- $x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \leq y$

---

distributive lattice satisfies

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

---

any embedding $P \hookrightarrow 2^{[n]}$

so $\leq, \vee, \wedge$ map to $\subseteq, \cup, \cap$

reveals P to be a ~~a~~ distributive lattice

(set intersection and union distribute over each other)

Fund Thm finite distributive lattice is precise description of ~~also~~ an embedding

giving converse.

$$\boxed{\underline{\text{Thm}} \quad L \text{ finite distributive lattice} \Rightarrow \exists! \text{ finite poset } P \text{ so } L \cong J(P)}$$

$J(P)$ = poset of order ideals of P

$L \cong J(P) \hookrightarrow 2^P$

is distinguished family of subsets of P

$x \vee y$ think $\cup$ how do they <u>join</u>?

$x \wedge y$ think $\cap$ where do they <u>meet</u>?

---

proof

$x \in L$ is <u>join-irreducible</u> $x \neq y \vee z$ for $x > y$, $x > z$

define P = induced subposet of join-irreds of L

want to show $J(P) \cong L$

$$\begin{tikzcd} L \arrow[r, bend left, hook, "\emptyset : x \mapsto I_x"] & J(P) \arrow[l, bend left, "I \mapsto x = \vee I"] \end{tikzcd}$$

where I_x = order ideal in P, $\{y \in P \mid y \leq x\}$

(all "atoms" underneath x)

$\vee I$ = join (in L) of all $y \in I$

(least x sitting over "atoms" in I)

want to show: $I = I_x$, $x = \vee I$ to show $\emptyset$ surjective: $\square$

$\emptyset$ is injective: $\square$

OK $\boxed{\emptyset : x \mapsto I_x \quad \text{is injective:}}$

every $x \in L$ is join of join-irred $y \leq x$

I_x is join-irred $y \leq x$

so I_x determines x.

induct on $x = y \vee z$, $y < x$, $z < x$

$\boxed{\text{Every } I \in J(P) \text{ is } I_x, \text{ some } x \in L}$

actually, $x = \bigvee I$

$I \subseteq I_x$ is clear (I_x is like closure of I but is it different?)

$I \supseteq I_x$?

If $z \in I_x$ want to show $z \in I$:

we know $\bigvee I = \bigvee I_x = x$ (in L)

apply $\wedge z$ and distributive laws:

$$\bigvee \{ y \wedge z \mid y \in I \} = \underbrace{\bigvee \{ y \wedge z \mid y \in I_x \}}_{z}$$

(P is just a poset, all $\vee, \wedge$ computations in L)

$\bigvee \{ y \wedge z \mid y \in I_x \}$ where $z \in I_x$

$y \wedge z \leq z$ for all y

$z \wedge z = z$ is one element of join

$\Rightarrow \quad = z$

$\bigvee \{ y \wedge z \mid y \in I \}$

certainly $\leq z$

because z is join-irred in L, at least one elem of join must be z

$\Rightarrow \quad y \wedge z = z$ for some $y \in I$

$\Rightarrow \quad$ ~~$y \wedge z$~~ $z \leq y$, $y \in I$ order ideal

$\Rightarrow z \in I$ //

We'll continue to pick up defs, structure theory of posets as we use it...

3.5 Chains in distributive lattices

relations between P and $J(P)$ (P finite)

$$\boxed{\#\{k \text{ elem order ideals of } P\} = \#\{\text{elems of } J(P), \text{ rank } k\}}$$

rank function $\rho : P \to \{0, 1, \ldots, n\}$

$\rho(x) = 0$ for minimal elems

$\rho(y) = \rho(x) + 1$ if y covers x

every maximal chain length $n \Leftrightarrow$ ~~$\exists$ rank function~~ $\Leftrightarrow$ P graded

$\Rightarrow \exists$ rank fn

i.e. $J(P)$ graded by ~~ρ~~ $\rho(I) = \# I$

$$\boxed{\#\{k \text{ elem antichains of } P\}_{k \geq 1} = \#\{I \in J(P) \mid I \text{ covers } \underbrace{\text{exactly } k \text{ elems}}_{\text{remove any of } k \text{ generators of } I}\}}$$

3.5.1 Prop P finite poset $m \geq 0$

Quantities equal:

(a) # order preserving $\sigma : P \to m$ (linearize w/ ties)

($\# \mathrm{Hom}(P, m)$ in $\mathcal{Q}_{\Rightarrow}$)

$=$ (b) # ~~multi~~ multichains $\hat{0} = I_0 \leq \ldots \leq I_m = \hat{1}$ in $J(P)$

$=$ (c) ~~card~~ $\# J(P \times m\text{-}1)$

In general $f : P \to Q$ induces $f^* : J(Q) \to J(P)$

pullback of order ideal is order ideal

(in $\mathcal{Q}_{\Rightarrow}$)

so unique chain

$I \subset Q$ order ideal

$x \in f^{-1}(I)$, $y \leq x$

$\Rightarrow f(y) \leq f(x) \Rightarrow f(y) \in I$

$\Rightarrow y \in f^{-1}(I)$

so order-preserving maps pull back order ideals

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Given $\sigma : P \to m$

look at unique maximal chain $\hat{0} = \emptyset \lneq I_1 \lneq I_2 \lneq \ldots \lneq I_m = \hat{1}$ in $J(m)$

pulls back to multichain in $J(P)$ which recovers σ

so (a) $\Leftrightarrow$ (b)

$I \subset P \times m-1 \qquad I = \{(x,j) \mid x \in I_{m-j}\}$

product structure "tags" a flag of order ideals

equivalent data

3.5.2 Prop

(a) # surjective $\sigma : P \to m$

= (b) # chains $\hat{0} = I_0 < I_1 < \ldots < I_m = \hat{1}$ in $J(P)$

same idea.

if $\#P = n$, these are linear extensions of P

$e(P)$ = # extensions = # maximal chains if $J(P)$