① comb class
Thurs 31 Jan 02

<u>descents</u> and <u>rising sequences</u>

$S = \{s_1, s_2\} = \{2,4\}$ descent set, $n=6$

$\underbrace{s_0=0,\ s_1=2}_{s_1-s_0=2},\ \underbrace{s_2=4}_{s_2-s_1=2},\ \underbrace{s_3=n=6}_{s_3-s_2=2}$

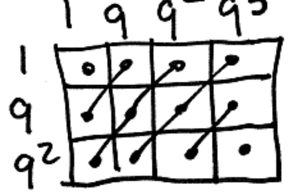
$\binom{n}{s_1, s_2-s_1, \ldots, n-s_k} = \binom{6}{2,2,2} = \dfrac{\cancel{6}^{3}\cdot 5\cdot \cancel{4}^{2}\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 90$

smaller example? $\binom{5}{2,2,1} = \dfrac{5\cdot\cancel{4}\cdot 3\cdot 2}{\cancel{2}\cdot\cancel{2}} = 30$

$\boxed{\binom{4}{2,1,1} = \dfrac{4\cdot 3\cdot\cancel{2}}{\cancel{2}} = 12}$

$S = \{s_1, s_2\} = \{2,3\}$
$n = 4$

$(4)(3) =$



$= 1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5$

so 12 solutions to $\binom{4}{2,1,1}$ have this distribution of inversions

multiset perms of $\{1,1,2,3\}$

0	1123	1234	1234
1	1213	1324	1324
1	1132	1243	1243
2	2113	3124	2314
2	1231	1342	1423
2	1312	1423	1342
3	2131	3142	2413
3	1321	1432	1432
3	3112	4123	2341
4	2311	3412	3412
4	3121	4132	2431
5	3211	4312	3421
		perm	inverse

$\beta_n(S,q) = q^3 + q^4 + q^5$

~~check using Mathematica?~~

don't be a wuss, do it by hand

recall $\binom{n}{a_1,\dots,a_m} = \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\dots-a_{m-1}}{a_m}$

has advantage in keeping computation integral/polynomial (no denominators)

$\binom{4}{2\ 1\ 1}$

$$\det\begin{bmatrix} \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\ 1 & \binom{2}{1} & \binom{2}{2} \\ 0 & 1 & \binom{1}{1} \end{bmatrix}$$

recall $\begin{bmatrix} a_{01} & a_{02} & a_{03} \\ 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \end{bmatrix}$ $-a_{02}a_{23}$ skips first descent

$$\begin{bmatrix} 6 & 4 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \det = 3 \ ✓$$

$$\cancel{\begin{bmatrix} 6 & 4 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 4 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = +3 \ ✓}$$

$$\binom{4}{2}_q = \frac{123321}{11} = 11211$$

$$\begin{array}{r} 11211 \\ 11211 \\ \hline 123321 \end{array} ✓$$

$123321 = q^5+2q^4+3q^3+3q^2+2q+1$

$$\begin{bmatrix} 11211 & 1111 & 1 \\ 1 & 11 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 11211 & 1110 & 0 \\ 1 & 10 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 11100 & 0 & 0 \\ 1 & 10 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

① = ① − 111②

$\det = 111000$
$= q^5+q^4+q^3$ ✓(!)

③ comb class
THUR 1/31/02

I saw this first shuffling w/ Persi Diaconis:

$1\ 2\ 3\ 4\ 5\ 6$

$1\ 2\ 3\ 4 \mid 5\ 6$ cut deck into 2 packets

$\pi = 1\ 5\ 2\ 3\ 6\ 4$ interleave two runs together

(using here $1\,2\,1\,1\,2\,1 \in \mathcal{S}(M)$, $M = \{1^4, 2^2\}$)

$\pi^{-1} = 1\ 3\ 4\ \underline{6}\ 2\ 5$ ← descent

a-shuffle, cut deck into a packets
all possible ways of interleaving equally likely

m 2-shuffles = one $a = 2^m$ shuffle

given π, odds of arising from shuffling is $\binom{a+n-r}{n} \Big/ a^n$
where $r = |D(\pi^{-1})| + 1$
$= \#$ rising sequences in π

e.g. $\pi = 1\,4\,2\,3\,5$ $\quad a=3,\ n=5,\ r=2$ $\quad \binom{a+n-r}{n} = \binom{3+5-2}{5} = 6$

<table>
<tr><th></th><th></th><th>multiset</th><th>$\pi \in \mathcal{S}(M)$</th></tr>
<tr><td>1 4 2 3 5</td><td>|1 2 3|4 5</td><td>2 2 2 3 3</td><td>2 3 2 2 3</td></tr>
<tr><td>1 4 2 3 5</td><td>1|2 3|4 5</td><td>1 2 2 3 3</td><td>1 3 2 2 3</td></tr>
<tr><td>1 4 2 3 5</td><td>1 2|3|4 5</td><td>1 1 2 3 3</td><td>1 3 1 2 3</td></tr>
<tr><td>1 4 2 3 5</td><td>1 2 3||4 5</td><td>1 1 1 3 3</td><td>1 3 1 1 3</td></tr>
<tr><td>1 4 2 3 5</td><td>1 2 3|4|5</td><td>1 1 1 2 3</td><td>1 2 1 1 3</td></tr>
<tr><td>1 4 2 3 5</td><td>1 2 3|4 5|</td><td>1 1 1 2 2</td><td>1 2 1 1 2</td></tr>
</table>

$a-1$ dividers
$r-1$ forced choices
$\Rightarrow a-r$ free dividers
fit among n cards

Switch topics, graph colorings as inclusion-exclusion

"Matroid theory"

Can use any abelian group, but simplest to choose field $\mathbb{F}_q$

(Color vertices of graph G using colors $\in \mathbb{F}_q$

no two neighboring vertices have same color)

$\Longleftrightarrow$ choose initial color for initial vertex

"constant of integration"

+ choose (difference) function $x : E \to \mathbb{F}_q$ on (oriented) edges of G

- sum to zero around each cycle
- never zero on any edge

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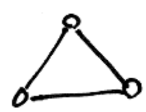
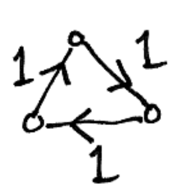
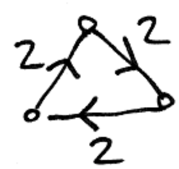
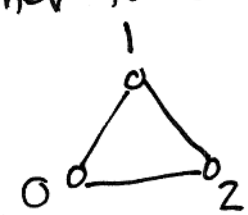
ex: $q = 3$ $G =$ 

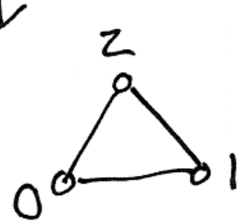
$x =$ 

or 

$\Downarrow$

set UL corner to 0





(mod 3)

given a spanning tree $T \subset G$, $x : E \to \mathbb{F}_q$

arbitrary on edges of T, forced on remaining edges

let $r = \#$ edges in T $n = \#$ edges in G

$V \subset \mathbb{F}_q^n$ $\dim V = r$

edge fns $\Sigma = 0$ around cycles

each edge $e \in E$ induces a hyperplane $H_e \subset V$ of forbidden colorings

want to count $V \setminus \{H_e \mid e \in E\}$ $\chi(q)$ is this count

given any subset $U \subset E$,

rank $r(U)$ is # edges in spanning forest in U

$= \#$ indep conditions imposed on V

$$\boxed{\dim \bigcap_{e \in U} H_e = r - r(U)}$$

inclusion-exclusion count $\iff$ chromatic polynomial characteristic polynomial

$$\boxed{\chi(q) = \sum_{U \subseteq E} (-1)^{U} q^{r - r(U)}}$$

<u>matroid theory</u> is formalization of such situations

set E together with known minimal dependencies

(+ conditions to make realistic) $\iff$ rank function

$\iff \ldots$

redo 2.3 <u>permutations with restricted position</u>

$\pi \in S_n \quad G(\pi) = \{(i, \pi(i)) \mid i = 1..n\} \subsetneq [n] \times [n]$

"graph of π"

$B \subseteq [n] \times [n]$ "board" of forbidden positions

For $j = 0..n$, $\boxed{N_j = \#\{\pi \in S_n \mid j = \#(B \cap G(\pi))\}}$

generating function $\boxed{N_n(x) = \sum_{j=0}^{n} N_j x^j}$

$N_n(0) = N_0 = \#\pi$ avoiding B entirely

$N_n(1) = n! = \#\pi \in S_n$

also define <u>rook polynomials</u>

$r_k = \#$ k-subsets of B, no common coords

$= \#$ non-attacking positions of k rooks on B

2.3.1 Theorem

$$\boxed{N_n(x) = \sum_{k=0}^{n} r_k (n-k)! \, (x-1)^k}$$

in particular $\boxed{N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)!}$

apply this to π w/ no fixed points

$$B = \{ (i,i) \mid i = 1..n \}$$

$$r_k = \binom{n}{k}$$

$$N_0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= \sum (-1)^k \frac{n!}{k!} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx n!/e$$

<u>first proof</u> count two ways # pairs (π, C)

$\pi \in S_n$, C k-subset of $B \cap G(\pi)$

1) for each j, pick π in N_j ways so $j = \#(B \cap G(\pi))$

pick C in $\binom{j}{k}$ ways

get $\boxed{\sum_{j=0}^{n} \binom{j}{k} N_j}$

2) choose C in r_k ways, extend to choice of π in $(n-k)!$ ways

get $\boxed{r_k (n-k)!}$

form generating function, summing over k using y^k

$$\boxed{\sum_{j=0}^{n} (y+1)^j N_j = \sum_{k=0}^{n} r_k (n-k)!\, y^k}$$

set $y = x-1$, getting

$$\boxed{N_n(x) = \sum_j N_j x^j = \sum_{k=0}^{n} r_k (n-k)! (x-1)^k}$$

second proof x positive integer

$\boxed{\sum_{j=0}^{n} N_j x^j}$ counts # ways placing n non-attacking rooks on $[n] \times [n]$,

then labeling each rook on B from $\{1, .., x\}$

$\boxed{\sum_{k=0}^{n} r_k (n-k)!\, (x-1)^k}$ counts # ways

place k nonattacking rooks on B

label from $\{2, .., x\}$

extend $(n-k)!$ ways to n nonattacking rooks

label each w/ 1

bijection for each $x \implies$ polynomials agree

(These are really the same proof)

<u>Problème des ménages</u> long history of unsatisfactory solutions

$\# \pi \in S_n$ so $\pi(i) \neq i, i+1 \pmod{n}$ for $\forall i$

[seat $n \geq 3$ married couples so M-F alternate
no couple sits adjacent]

(from online search; how is this equivalent?)

X	X				
	X	X			
		X	X		
			X	X	
				X	X
X					X

r_K = # ways choosing
K nonadjacent points
from 2n in a circle

$B = \{ (i,i), (i, \underbrace{i+1}_{\text{mod } n}) \mid i = 1..n \}$

<u>lemma</u> K from m in circle $= \frac{m}{m-K} \binom{m-K}{K}$

<u>first proof</u> $f(m,K)$ = desired #
$g(m,K)$ = ... and mark one point not chosen

$$\boxed{g(m,K) = (m-K)\, f(m,K)}$$ (explains denominator)

~~also~~ mark a point, m ways
now choose K nonadjacent from linear array m−1

place m−1−K 'not chosen' points in a line
m−K slots available for K chosen

$$\boxed{g(m,K) = m \binom{m-K}{K}}$$ //

2nd proof label points $1, 2, \ldots, m$ in cyclic order

color k red, no two adjacent

1 isn't red:

place $m-k$ uncolored points on a circle

label one '1'

insert k red points into $m-k$ gaps, $\binom{m-k}{k}$ ways

1 is red:

place $m-k+1$ uncolored pts in circle

color one red and label 1

insert $k-1$ red points into $m-k-1$ allowed gaps

$$\binom{m-k}{k} + \binom{m-k-1}{k-1}$$

$$= \left[1 + \frac{k}{m-k}\right]\binom{m-k}{k} = \frac{m}{m-k}\binom{m-k}{k} \quad //$$

2.3.5. Corollary $N_n(x)$ for  is

$$N_n(x) = \sum_{k=0}^{n} r_k (n-k)!\,(x-1)^k$$

$$= \sum_{k=0}^{\infty} \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!\,(x-1)^k$$

so

$$\boxed{N_0 = \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!\,(-1)^k}$$

hard formula,
broken into easy steps