① Tues Comb class

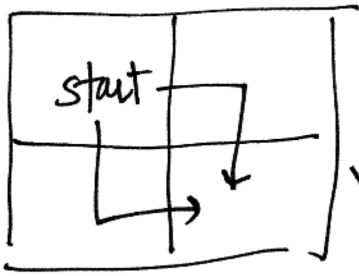
q-analogue

$(i) = 1+q+\dots+q^{i-1}$

$(i)! = (1)(2)\cdots(i)$

$\binom{n}{a_1,\dots,a_m}_q = \frac{(n)!}{(a_1)!\cdots(a_m)!}$

sets $\longrightarrow$ subspaces of $\mathbb{F}_q^n$



$\downarrow$ refined ~~statistics~~ ~~partition of~~ structure on permutations

One make q-analogues even if only get partway, but goal always to see everything at once.

E.g. $\binom{n}{k} \Rightarrow \binom{n}{k}_q$

k-subsets of n	k-subspaces of $\mathbb{F}_q^n$
inversions of $\pi \in S(M)$	Schubert cells of Grassmanian

✓ we did this

$M = \{1^k, 2^{n-k}\}$

this generalizes for $a_1+\dots+a_m = n$

$\binom{n}{a_1,\dots,a_m} \Rightarrow \binom{n}{a_1,\dots,a_m}_q$

ordered $a_1,\dots,a_m$ partitions of $1..n$	flags in $\mathbb{F}_q^n$ dim $a_1, a_1+a_2, \dots$
inversions of $\pi \in S(M)$ $M = \{1^{a_1},\dots,m^{a_m}\}$	Schubert cells of flag variety

$\Longleftrightarrow$ ex: $\# = \binom{n}{a_1,\dots,a_m}_q$

$\Longleftrightarrow$ ex: again similar to Grassmanian

We'll do most of this, leave rest as exercise

$i(\pi)$ = # inversions of $\pi \in S_n$ or $S(M)$, $M = \{1^{a_1}, \dots, m^{a_m}\}$

= # out of order pairs $i<j$ and $b_i > b_j$

$$\boxed{\sum_{\pi \in S_n} q^{i(\pi)} = (n)!}$$

proof: bijection between integer data $(a_1, \dots, a_n)$ $\quad 0 \le a_i \le n-i$

and perm $\pi = b_1 b_2 \cdots b_n \in S_n$

construction: insert $n, n-1, n-2$ progressively into π as <u>word</u>

insert $n-i$ so it has a_{n-i} elems to left

all inversions ending in $n-i$ already created, rest of constr'n has no effect on $(j, n-i)$

$a_i = \# j$ to left of i with $j > i$

$(a_1, \dots, a_9) = (1, 5, 2, 0, 4, 2, 0, 1, 0)$

$\pi = 417396285$

$I(\pi) = (a_1, \dots, a_n) \Rightarrow i(\pi) = a_1 + \dots + a_n$

$$\sum_{\pi \in S_n} q^{i(\pi)} = \sum_{a_1=0}^{n-1} \cdots \sum_{a_n=0}^{0} q^{a_1 + \dots + a_n}$$

$$= \left(\sum_{a_1=0}^{n-1} q^{a_1}\right) \cdots \left(\sum_{a_n=0}^{0} q^{a_n}\right)$$

$= (n)!$ as desired.

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generalizes to $$\boxed{\sum_{\pi\in S(m)} q^{i(\pi)} = \binom{n}{a_1,\dots,a_m}_q}$$

(each $a_i = 1$ gives $(n)!$)
ordinary case $\pi \in S_n$

second proof from book

$$\phi : S(m) \times S_{a_1} \times \dots \times S_{a_m} \longrightarrow S_n$$

$$(\pi_0, \pi_1, \dots, \pi_m) \longmapsto \pi$$

converting a_i i's in π_0 to $(a_1+a_2+\dots+a_{i-1})+1, \dots$
$(a_1+\dots+a_{i-1})+a_i$

in order specified by π_i

ϕ is bijection
$$i(\pi) = i(\pi_0) + i(\pi_1) + \dots + i(\pi_m)$$

so $$\Big(\sum_{\pi\in S(m)} q^{i(\pi)}\Big)(a_1)! \cdots (a_m)! = (n)! \qquad //$$

Now Schubert cells take $G(2,4)$

$\begin{bmatrix} 0&0&1&0\\0&0&0&1\end{bmatrix}$ 1122 $i(\pi)=0$

$\begin{bmatrix} 0&1&*&0\\0&0&0&1\end{bmatrix}$ 1212 $i(\pi)=1$

$\begin{bmatrix} 1&*&*&0\\0&0&0&1\end{bmatrix}$ 2112 $i(\pi)=2$

$\begin{bmatrix} 0&1&0&*\\0&0&1&*\end{bmatrix}$ 1221 $i(\pi)=2$

$\begin{bmatrix} 1&*&0&*\\0&0&1&*\end{bmatrix}$ 2121 $i(\pi)=3$

$\begin{bmatrix} 1&0&*&*\\0&1&*&*\end{bmatrix}$ 2211 $i(\pi)=4$

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

partitions points in $G(2,4)$
according to Schubert cell

0   1   2   1+1   2+1   2+2

another way of phrasing exactly same thing

$$\sum_m \begin{pmatrix} \#\text{partitions of } m \\ \text{at most } k \text{ parts} \\ \text{biggest} \leq n-k \end{pmatrix} q^m = \binom{n}{k}_q$$

---

generalize: $\pi \in \mathcal{S}(M)$, $M = \{1^{a_1}, \ldots, m^{a_m}\}$

gives instructions for creating row-reduced basis for flag,

all same idea.

exercise to chase through...

$S = k[x_0, \ldots, x_n]$ $\mathbb{P}^n = \operatorname{Proj} S$

$$\sum_m \dim S_m t^m = \frac{1}{(1-t)^n} = \sum \binom{n+m}{n} t^n$$

what is q-analogue

$$\boxed{\sum_m \dim \hat{\mathfrak{S}}_m t^m = \sum \binom{n+m}{n}_q t^n} \quad ?$$

I don't know, there must be one, probably useful.

So <u>now</u> we can turn page on q-analogue of our count

$D(\pi)$ descent set of $\pi \in S_n$

$$\alpha_n(S) = \#\{\pi \mid D(\pi) \subseteq S\}$$

$$\beta_n(S) = \#\{\pi \mid D(\pi) = S\}$$

$$\alpha_n(S) = \sum_{T \subseteq S} \beta_n(T)$$

$$S = \{s_1, \dots, s_k\} \subset \{1, \dots, n-1\}$$

$$\Longrightarrow \boxed{\beta_n(S) = n! \det\left[1/(s_{j+1}-s_i)! \right]_{i,j=0..k+1} \qquad s_0 = 0,\ s_{k+1} = n}$$

so q-analogue is clear:

$$\boxed{\beta_n(S) = (n)! \det\left[1/(s_{j+1}-s_i)! \right]_{i,j=0,..k+1}}$$

q-analogue (k)!

but what is this counting (everything now poly in q)?

let $M = \{1^{s_1}, 2^{s_2-s_1}, \dots, (k+1)^{n-s_k}\}$

$$\boxed{\sum_{\sigma \in S(M)} q^{i(\sigma)} = \binom{n}{s_1, s_2-s_1, .. s_k-s_{k-1}, n-s_k}_q}$$

as before

So map $\mathcal{S}(M) \hookrightarrow \mathcal{S}_n$

from before

$$\mathcal{S}(M) \times \mathcal{S}_{a_1} \times \dots \mathcal{S}_{a_m} \to \mathcal{S}_n$$
$$\pi_0 \times id \times \dots \times id$$

(<u>refine</u> a_i i's to $i, i+\varepsilon, i+2\varepsilon, \dots$
then renumber 1 through n)

since $i(id) = 0$, injection preserves inversions

So we've proved

$$\sum_{\substack{\pi \in \mathcal{S}_n \\ D(\pi) \subseteq S}} q^{i(\pi)} = \binom{n}{s_1, s_2 - s_1, \dots, s_k - s_{k-1}, n - s_k}_q$$

so
$$\boxed{\beta_n(S,q) = \sum_{\substack{\pi \in \mathcal{S}_n \\ D(\pi) = S}} q^{i(\pi)} = \underbrace{(n!) \det\left[\tfrac{1}{(s_{j+1} - s_i)} \right]_{i,j=0..k}}_{q\text{-analogue}}}$$

set $q=1$ of course aggregates this count to earlier one.

$$\boxed{\text{Q: What's the subspace interpretation?}}$$

<u>new topic(!?)</u> 2.3 permutations w/ restricted position.

B $\quad [n] = \{1, 2, \ldots, n\}$

$B \subseteq [n] \times [n]$ a "board"

count $\pi \in S_n$ so for each i, certain $\pi(i)$ disallowed

<u>derangement problem:</u> $\pi(i) \neq i$ $\quad B \subseteq [n] \times [n]$ is diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we want to generalize. (convention $\pi = \pi(1)\pi(2)\ldots\pi(n)$)

(I <u>think</u> this is stanley's also)

graph $G(\pi) = \{(i, \pi(i)) \mid i = 1..n\}$

tatoo into brain:

$$N_j = \#\{\pi \in S_n \mid j = \#(B \cap G(\pi))\}$$

number of forbidden positions in π

r_k = # k-subsets of B, no two elems common coord

= # nonattacking positions, k rooks on B

$\pi \Leftrightarrow G(\pi)$ is a position of n nonattacking rooks on $n \times n$ chessboard

(original derangement problem, N_0 where B was diagonal $\{(i,i)\}$)

make generating function of $N_0, \ldots, N_n$:

$$N_n(x) = \sum_j N_j x^j \qquad \left(\begin{array}{l} x \text{ is dummy var} \\ \text{carrying data} \end{array}\right)$$

Theorem 2.3.1

$$\boxed{N_n(x) = \sum_{k=0}^{n} r_k (n-k)! \, (x-1)^k}$$

so $\boxed{N_0 = N_n(0) = \sum_{k=0}^{n} (-1)^k r_k (n-k)!}$

and if B is just diagonal, $r_k = \binom{n}{k}$

recover original count $n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \approx n!/e$

So why??

<u>first proof</u>

$$\boxed{\begin{array}{ll} C_k = \#\text{ pairs } (\pi, C) & \pi \in S_n \\ & C \text{ is } k\text{-subset of} \\ & B \cap G(\pi) \end{array}}$$

(nec. nonattacking because)

for each j, choose π in N_j ways so $j = \#(B \cap G(\pi))$
then choose C in $\binom{j}{k}$ ways

$$\Rightarrow \quad C_k = \sum_j \binom{j}{k} N_j$$

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or choose C in r_k ways, extend to π in $(n-k)!$ ways,

so $\boxed{c_k = r_k (n-k)!}$ tricky, of course doesn't matter if remaining choices hit B or not.

submerged sieve method !!

$$\Rightarrow \quad \boxed{\sum_j \binom{j}{k} N_j = r_k (n-k)!} \qquad \text{equating formulas for } c_k$$

$\Downarrow$ have dummy var y carry each k as power

$$\boxed{\sum_j (y+1)^j N_j = \sum_k r_k (n-k)!\, y^k}$$

$\Downarrow$ set $y = x-1$

$$\boxed{\sum_j x^j N_j = \sum_k r_k (n-k)!\, (x-1)^k} \qquad \boxed{\text{yikes!!}}$$

---

second proof assume $x > 0$.

left side counts # ways placing n rooks on $[n]\times[n]$, labeling each rook on B with elem of $[x]$

alternatively, place k nonattacking rooks on B, label w/ $\{2, .., x\}$

place $n-k$ nonattacking add'l on $[n]\times[n]$ in $(n-k)!$ ways,

label new rooks with 1

bijection for enough values of x $\Rightarrow$ formula holds

(problème des ménages) $M(n) = \#$ perms $\pi \in S_n$
so $\pi(i) \neq i, i+1 \pmod n$
for all i

$$B = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix} \quad \text{seek } N_0$$

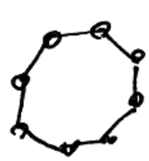
r_k = choose k points, no two consecutive, from $2n$ in a circle

$\boxed{k \text{ from } m \text{ is } \frac{m}{m-k}\binom{m-k}{k}}$

lemma

$$\Rightarrow N_n(x) = \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!\,(x-1)!$$

(cottage industry of figuring $N_n(x)$ by computing instead r_k)

second proof label points $1, 2, \ldots, m$ clockwise.
color k red, no two consecutive.
This is just chromatic polynomial for  m-gon at

NO! because we're refined to how many of each color gets used.
Interesting refinement for any graph.

back to second proof

color k red out of m, no two consecutive.

1 isn't red

insert k red in $\binom{m-k}{k}$ ways

place $m-k$ uncolored pts
label one 1

or, 1 is red   place $m-k+1$ in circle, color 1 red,

insert $\binom{m-k-1}{k-1}$ ways into allowed spaces

$$\frac{m}{m-k}\binom{m-k}{k} = \binom{m-k}{k} + \binom{m-k-1}{k-1}$$

$$\left(1+\frac{k}{m-k}\right)\binom{m-k}{k} \quad ✓$$

---

move onward...

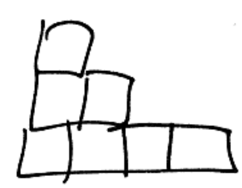
Ferrers Boards, rook polys get special

given $0 \le b_1 \le \dots \le b_m$

Ferrers Board of shape $(b_1,\dots,b_m)$ is

$$B = \{(i,j) \mid 1 \le i \le m,\ 1 \le j \le b_i\}$$

technical convenience to allow $b_i = 0$.



Thm

$$\sum r_k (x)_{m-k} = \prod_1^m (x+s_i) \quad \text{where } s_i = b_i - i + 1$$

rook polynomial

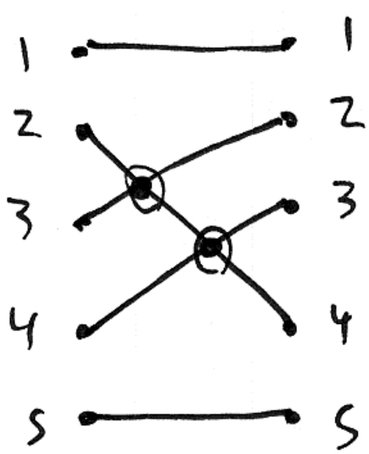
~~descents~~

Inversions same for π, π^{-1}:

Visual proof: graph the permutation

$\pi = 1\,4\,2\,3\,5$

$\pi^{-1} = 1\,3\,4\,2\,5$



$i(\pi) = 2$

read one way as π, other way as π^{-1}

$\#$ crossings $= i(\pi) = i(\pi^{-1})$