comb class 1/24/02 THURS

~~#24~~

subsets $U, V \subseteq S$

$f_=(U)$ count w/ exactly properties in U

$f_\geq(U)$ count w/ at least properties in U

ζ zeta function

$$\zeta = \sum_{U \subseteq V} [U,V]$$

where $[U,V]$ is map $e_V \mapsto e_U$

$$[U,V][W,X] = \begin{cases} [U,X] & V=W \\ 0 & V \neq W \end{cases}$$

$$\mu = \sum_{U \subseteq V} (-1)^{|V-U|} [U,V]$$

$$\mu\zeta = \left(\sum_{U \subseteq T} (-1)^{|T-U|} [U,T]\right)\left(\sum_{T \subseteq V} [T,V]\right)$$

$$= \sum_{U \subseteq V} \underbrace{\left(\sum_{T \text{ so } U \subseteq T \subseteq V} (-1)^{|T-U|}\right)}_{\substack{1,\ U=V \\ 0,\ U \subset V}} [U,V] = Id$$

$$f_\geq = \zeta f_= \quad \text{so} \quad f_= = \mu f_\geq$$

generalize to poset P

$$\zeta = \sum_{a \le b} [a,b] \qquad \mu = \zeta^{-1}$$

$$f_{\ge} = \zeta f_{=}$$

$$\Rightarrow f_{=} = \mu f_{\ge}$$

~~#~~ $\ge$ ~~too~~

$f_{=}(a)$ count associated with element a of P

$f_{\ge}(a)$ " " elements $\ge a$ in P

$$\mu = \sum_{a \le b} \mu_{ab} [a,b]$$

where $\mu_{aa} = 1$

$$\mu_{ab} = \sum_{\cancel{\text{chains of length } i}} (-1)^i (\#\text{chains length } i)$$

$$\mu_{ab} = \tilde{\chi}(\Delta((a,b)))$$

$$a = c_0 < \underbrace{c_1 < \ldots < c_i}_{\text{length } i} = b$$

$$\mu\zeta = \left(\sum_{a \le c} \mu_{ac}[a,c]\right)\left(\sum_{c \le b} [c,b]\right)$$

$$= \sum_{a \le b} \Big(\underbrace{\sum_{c \text{ so } a \le c \le b} \mu_{ac}}_{\substack{\text{want } 1,\ a=b \\ 0,\ a<b}} \Big) [a,b] = Id$$

$\boxed{\checkmark}$

$\mu_{aa} = +1$ pairs with $a = c_0 < c_1 = c$, -1 for each $c < b$

$a = c_0 < c_1 < \ldots c_i = c$ $(-1)^i$ pairs with $(-1)^{i+1}$

$a = c_0 < c_1 < \ldots < c_i = c < \cancel{b}_{i+1} = !$

So two equivalent ways to compute Poincare series of

$$I = (b^2, bc, c^2) < k[a,b,c,d] = S$$

$$\boxed{\sum \dim(S/I)_m t^m}$$

$$\sum \dim(S)_m t^m = \frac{1}{(1-t)^4}$$

sum all monomials in S:

$$(1+a+a^2+\ldots)(1+b+b^2+\ldots)(1+c+c^2+\ldots)(1+d+d^2+\ldots)$$

$$= \left(\frac{1}{1-a}\right)\left(\frac{1}{1-b}\right)\left(\frac{1}{1-c}\right)\left(\frac{1}{1-d}\right)$$

now $a=b=c=d=t \Rightarrow \frac{1}{(1-t)^4} = \sum \left(\begin{matrix}\#\text{ monoms} \\ \deg m \\ \text{in } a,b,c,d\end{matrix}\right) t^m$

should also know "bars & stars" argument

$$\#\text{monoms deg } d \text{ in } x_0,\ldots,x_n = \binom{d+n}{n}$$

$d+n$ slots
n dividers makes $n+1$ bins
remaining slots gives monoms

a^2	** | |	$\binom{4}{2}=6$
ab	* | * |	
b^2	| ** |	
ac	* | | *	
bc	| * | *	
c^2	| | **	

Then prove by induction

$$\sum \binom{d+n}{n} t^d = \frac{1}{(1-t)^{n+1}}$$

$$\frac{1}{1-t}\left(\frac{1}{(1-t)^n}\right)$$

$$\binom{0+n-1}{n-1} + \binom{1+n-1}{n-1} + \ldots + \binom{d+n-1}{n-1} = \binom{d+n}{n}$$

$$\sum_m \dim \Bbbk(b^2)_m t^m = \frac{t^2}{(1-t)^4} \text{, etc.}$$

inclusion/exclusion

$$\cancel{(1) - (b^2) - (bc) - (c^2) + (b^2)\cap(bc)}$$

(1)
$-(b^2)$
$-(bc)$
$-(c^2)$
$+(b^2)\cap(bc) = (b^2c)$
$+(bc)\cap(c^2) = (bc^2)$
$\left.\begin{array}{l} +(b^2)\cap(c^2) = (b^2c^2) \\ -(b^2)\cap(bc)\cap(c^2) = (b^2c^2) \end{array}\right\}$ cancel to 0

$$\frac{1-3t^2+2t^3}{(1-t)^4} = \frac{3}{(1-t)^2} - \frac{2}{(1-t)}$$

$$= 3\mathbb{P}^1 - 2\mathbb{P}^0$$

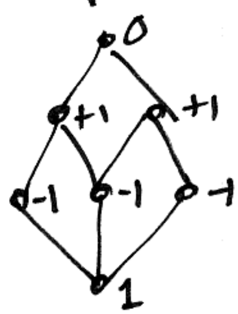
or directly using lcm poset

b^2c^2

$b^2c \quad bc^2$

$b^2 \quad bc \quad c^2$

$\hat{0}$



$\mu_{\hat{0},x}$

$$0 \leftarrow S/I \leftarrow S \xleftarrow{[b^2\ bc\ c^2]} S^3 \xleftarrow{\begin{bmatrix}-c & 0\\ b & -c\\ 0 & b\end{bmatrix}} S^2 \leftarrow 0$$

same idea, only linear algebra

<u>harder problem</u> <u>descents</u> in a permutation (ch 1)

$$\pi = a_1 a_2 \cdots a_n$$
$$D(\pi) = \{ i \mid a_i > a_{i+1} \} \subseteq \{1, \dots, n-1\}$$

(relevance: shuffling probabilities depend on $|D(\pi^{-1})|$)

$$\alpha(S) = \# \{ \pi \in S_n \mid D(\pi) \subseteq S \}$$
$$\beta(S) = \# \{ \pi \in S_n \mid D(\pi) = S \}$$

$$\alpha(S) = \sum_{T \subseteq S} \beta(T)$$ order reversed, but same-old same-old

$\alpha(S)$ easier to compute

$$S = \{ s_1, \dots, s_k \} \subset \{1, \dots, n-1\}$$

choose s_1 elems in order, first run $a_1 a_2 \cdots a_{s_1}$

$\binom{n}{s_1}$ ways $\qquad a_1 < a_2 < \cdots a_{s_1}$

choose $s_2 - s_1$ for next $a_{s_1+1} < \cdots a_{s_2}$

$\binom{n-s_1}{s_2-s_1}$ $\qquad a_1 \cdots a_{s_1} a_{s_1+1} \cdots a_{s_2}$

might <u>not</u> be a descent
can't be anywhere else,
so $D(\pi) \subseteq S$

$$\alpha(S) = \binom{n}{s_1}\binom{n-s_1}{s_2-s_1} \cdots \binom{n-s_k}{n-s_k}$$
$$= \binom{n}{s_1, s_2-s_1, \dots, n-s_k}$$ multinomial coef
k+1 teams of specified sizes

now what?

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T)$$

real work begins, what *is* this?

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trick involving determinant

$$\begin{vmatrix} a_{12} & a_{13} \\ 1 & a_{23} \end{vmatrix} = a_{12}a_{13} - a_{13}$$

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ 1 & a_{23} & a_{24} \\ 0 & 1 & a_{34} \end{vmatrix} = a_{12}a_{23}a_{34} - a_{13}a_{34} - a_{12}a_{24} + a_{14}$$

etc...

$s_1, s_2 - s_1, s_3 - s_2, \ldots n - s_k$

$\cancel{t_1 + t_2 + \ldots + t_{K+1}}$

$\cancel{t_1 + t_2 + \ldots + t_{K+1}}$

$\cancel{\binom{n}{t_1}\binom{n-t_1}{t_2} \cdots \binom{n-t_K}{t}}$

$\binom{n}{s_1}\binom{n-s_1}{s_2-s_1} \cdots \binom{n-s_K}{n-s_K}$

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trick involving determinant

$$\begin{vmatrix} a_{01} & a_{02} \\ 1 & a_{12} \end{vmatrix} = a_{01}a_{12} - a_{02}$$

$$\begin{vmatrix} a_{01} & a_{02} & a_{03} \\ 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \end{vmatrix} = \underbrace{a_{01}a_{12}a_{23}}_{\text{descents at } s_1, s_2} - \underbrace{a_{01}a_{13}}_{\text{only } s_1} - \underbrace{a_{02}a_{23}}_{\text{only } s_2} + \underbrace{a_{03}}_{\text{no descents}}$$

$$\binom{n}{s_1, s_2-s_1, n-s_2} = n! \cdot \frac{1}{(s_1)!} \cdot \frac{1}{(s_2-s_1)!} \cdot \frac{1}{(n-s_2)!}$$

e.g. $n=8$, $S=\{1,5\}$

$$\beta(S) = 8! \begin{vmatrix} \frac{1}{1!} & \frac{1}{5!} & \frac{1}{8!} \\ 1 & \frac{1}{4!} & \frac{1}{7!} \\ 0 & 1 & \frac{1}{3!} \end{vmatrix} = 217$$

(pretty cool.) converts exponential solution to polynomial soln

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"q-analogue" to most such computations.

"q-multinomial coefficient" $\quad a_1 + \cdots + a_m = n$

$$\binom{n}{a_1, \ldots, a_m}_q = \frac{(n)!}{(a_1)! \cdots (a_m)!} \qquad \binom{n}{k}_q = \binom{n}{k, n-k}_q$$

where $(k)! = (1)(2)\cdots(k)$

$(j) = 1 + q + q^2 + \cdots + q^{j-1}$

note restriction $q = 1$:

$(j) = j, \quad (k)! = k! \quad \binom{n}{a_1, \ldots a_m}_q = \binom{n}{a_1 \cdots a_m}$

(char 1)!

what does this mean for general q?

<u>Prop</u> # k-dim subspaces of $\mathbb{F}_q^n$ is $\binom{n}{k}_q = G(n,k)$

count ordered bases $(v_1, \ldots, v_k)$ in $\mathbb{F}_q^n$

$N = (q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$

$G(n,k)$ is number in question

$N = G(n,k)(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$

$$\Rightarrow \quad G(n,k) = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}$$

cancels out to $\dfrac{(1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1+q+\cdots q^{n-k})}{(1+\cdots+q^{k-1})\cdots(1)}$

$= \binom{n}{k}_q$

want to take wild guess what $\binom{n}{a_1,\ldots a_K}_q$ means?

"Gaussian polynomial"

$$\binom{n}{a_1,\ldots a_m}_q = \binom{n}{a_1}_q \binom{n-a_1}{a_2}_q \cdots \binom{a_m}{a_m}_q$$

and $$\binom{n}{K}_q = \binom{n-1}{K}_q + q^{n-K}\binom{n-1}{K-1}_q$$

$\binom{n}{0} = 1$ initial conditions

$$\frac{(n)(n-1)\cdots(n-K+1)}{(K)(K-1)\cdots(1)} = \frac{(n-1)(n-2)\cdots(n-K)}{(K)(K-1)\cdots(1)} + \frac{(n-1)(n-2)\cdots(n-K+1)}{(K-1)(K-2)\cdots(1)} q^{n-K}$$

$$1 = \frac{(n-K)}{(n)} + \frac{(K)}{(n)} q^{n-K}$$

$$\boxed{(n) = (n-K) + (K) q^{n-K}}$$

so always a polynomial (generalizes always an integer)

1.3.17 $M = \{1^{a_1}, \ldots, m^{a_m}\}$ multiset of card $n = a_1 + \cdots + a_m$

$\mathcal{S}(M)$ perms of multiset $b_1 b_2 \cdots b_n$

a_i copies of i each i

$i(\pi) = \#$ inversions of $\pi \in \mathcal{S}_n$

(b_i, b_j) is <u>inversion</u> of π if $i<j$ and $b_i > b_j$

\# out-of-order pairs

1.3.10

$$\sum_{\pi \in S_n} q^{i(\pi)} = \cancel{\quad} \quad (n)! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

why? $\quad I(\pi)$ = inversion table of π

integer sequence $(a_1, \ldots, a_n)$ $\quad 0 \le a_i \le n-i$
same data as permutation $\pi = b_1 b_2 \cdots b_n$

insert $n, n-1, n-2, \ldots$ progressively into word π

insert $n-i$ so it has a_{n-i} elements to left

note i already placed, and $0 \le a_{n-i} \le i$

$(a_1, \ldots a_9) = (1,5,2,0,4,2,0,1,0)$ $\quad$ 417396285

$a_i = \#\, j$ to left of i with $j > i$

$$I(\pi) = (a_1, \ldots a_n) \Leftrightarrow i(\pi) = a_1 + \cdots + a_n$$

$$\sum_{\pi \in S_n} q^{i(\pi)} = \sum_{a_1=0}^{n-1} \cdots \sum_{a_n=0}^{0} q^{a_1 + \cdots + a_n}$$

$$= \left(\sum_{a_1=0}^{n-1} q^{a_1}\right)\left(\sum_{a_2=0}^{n-2} q^{a_2}\right) \cdots \left(\sum_{a_n=0}^{0} q^{a_n}\right)$$

$= (n)!$ as desired

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generalizes to

$a_i = 1$ gives $(n)!$

$$\sum_{\pi \in S(M)} q^{i(\pi)} = \binom{n}{a_1, \ldots a_m}_q$$

(1.3.17 two proofs)

notice formally that if we can find $s(\pi)$ so

$$\sum_{\substack{\pi \in S_n \\ D(\pi) \subseteq S}} q^{s(\pi)} = \binom{n}{s_1, s_2 - s_1, \ldots, n - s_k}_q$$

we can mimic above reasoning to get a stronger result

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claim $s(\pi) = i(\pi)$, # inversions of π

$$\sum_{\sigma \in S(M)} q^{i(\sigma)} = \binom{n}{t_1, \ldots, t_{k+1}}$$

$t_1 = s_1$
$t_2 = s_2 - s_1$

$t_k = s_k - s_{k-1}$
$t_{k+1} = n - s_k$

$$M = \{1^{t_1}, \ldots (k+1)^{t_{k+1}}\}$$

given $\sigma \in S(M)$, build $\tau \in S_n$ by replacing t_1 1's by $1 \ldots s_1$

t_2 2's by $s_1 + 1, \ldots s_2$ in order

(τ is <u>shuffle</u> of $\{1, \ldots s_1\}, \{s_1 + 1, \ldots s_2\}, \ldots \{s_k + 1, \ldots n\}$)

$i(\sigma) = i(\tau)$ haven't changed # inversions

now set $\pi = \tau^{-1}$ $\quad$ τ is shuffle $\Leftrightarrow$ $D(\pi) \subseteq \{s_1, \ldots s_k\}$

$i(\tau) = i(\pi)$

$$\Longrightarrow \alpha_n(S, q) = \sum_{\substack{\pi \in S_n \\ D(\pi) \subseteq S}} q^{i(\pi)} = \binom{n}{s_1, \ldots, n - s_k}_q \qquad \beta_n(S, q) = \sum_{\substack{\pi \in S_n \\ D(\pi) = S_n}} q^{i(\pi)}$$

$$\beta_n(S, q) = (n)! \det\left[\frac{1}{(s_{j+1} - s_i)!} \right]$$

go back and prove 1.3.17

2nd proof

$$\phi : \mathcal{S}(M) \times \mathcal{S}_{a_1} \times \dots \times \mathcal{S}_{a_m} \longrightarrow \mathcal{S}_n$$

$$(\pi_0, \pi_1, \dots, \pi_m) \longmapsto \pi$$

converting a_i i's in π_0 to $a_1 + .. + a_{i-1} + 1, \dots + a_i$
in order specified by π_i

- ϕ is bijection,

$$i(\pi) = i(\pi_0) + i(\pi_1) + \cdots + i(\pi_m)$$

$$\Longrightarrow \quad \Big(\sum_{\pi \in \mathcal{S}(M)} q^{i(\pi)} \Big) (a_1)! \cdots (a_m)! = (n)!$$

1.3.19 $\quad p(j, k, n)$ partitions of n
into at most k parts
largest part $\leq j$

$$\sum_{n \geq 0} p(j, k, n) q^n = \binom{j+k}{j}_q$$

q-analogue of monoms deg k in $j+1$ vars given $1 \le a_1 < a_2 < \dots < a_k \le m$

$m = j+k$ look at k-dim subspaces of $\mathbb{F}_q^{\,m}$

look at unique row-echelon form for basis

$$k\left\{\underbrace{\begin{array}{ccccccc} 1 & * & 0 & 0 & * & 0 & * \\ & & 1 & 0 & * & 0 & * \\ & & & 1 & * & 0 & * \\ & & & & & 1 & * \end{array}}_{m}\right.$$

first $\neq 0$ entry of row i is col a_i

λ_i *'s in row i is $j - a_i + i$

$\lambda_1 \dots \lambda_k$ partition of some n into $\le k$ parts
largest part $\le j$

$$\binom{j+k}{k}_q = \sum_{\substack{\text{all } \lambda \\ \le k \text{ parts} \\ \text{largest part} \le j}} q^{|\lambda|} = \sum_{n \ge 0} p(j,k,n)\, q^n$$

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needs some justification !!