① Combinatorics Course
Tues, 22 Jan 02

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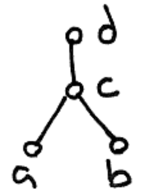
central object of study: <u>poset</u> or partially ordered set

P set w/ relation $\leq$ so

1) $x \leq x$
2) $x \leq y$, $y \leq x \Rightarrow x = y$
3) $x \leq y$, $y \leq z \Rightarrow x \leq z$

usually drawn $\begin{matrix} y \\ | \\ x \end{matrix}$ if $x \lneq y$ but no $x \lneq z \lneq y$

(y 'covers' x)



maximal chains are $a < c < d$, $b < c < d$

completes to total orders $a < b < c < d$,

$b < a < c < d$

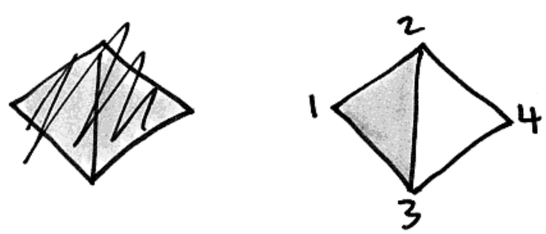
Def $X \subseteq 2^{\{1,\dots,n\}}$ simplicial complex on $\{1,\dots,n\}$

if $F \in X$, $G \subseteq F \Rightarrow G \in X$

(closed under subsets)

Think of as real topological space by gluing simplices

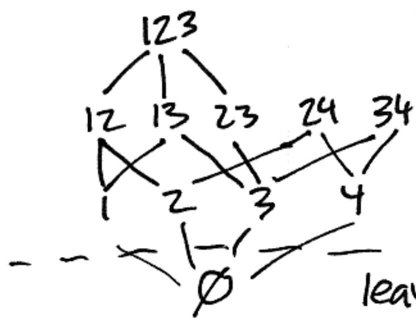
<u>geometric realization</u>



$X = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 24, 34, 123\}$

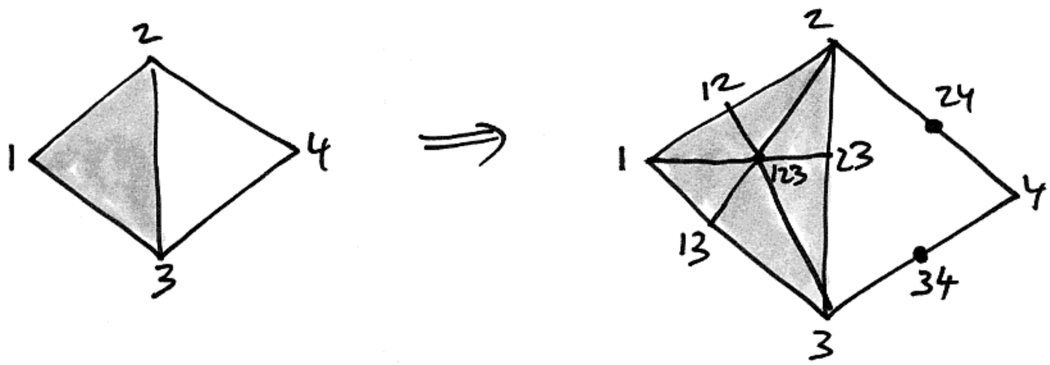
induces <u>face poset</u> of X, nontrivial faces of X ordered under inclusion

P_X



leave off empty set

Given X can form barycentric subdivision X'



faces ~~#~~ of X' are chains in P_X

maximal chains:

$$\begin{array}{ll} 2 < 24 & 1 < 12 < 123 \\ 4 < 24 & 2 < 12 < 123 \\ 3 < 34 & 2 < 23 < 123 \\ 4 < 34 & 3 < 23 < 123 \\ & 3 < 13 < 123 \\ & 1 < 13 < 123 \end{array}$$

Given any poset P can form order complex $\Delta(P)$

$$\Delta(P) \subseteq 2^P$$

$$F \in \Delta(P) \iff F \subset P \text{ is a (totally ordered) chain}$$

yields simplicial complex

For us, $\Delta(P_X) = X'$

- barycentric subdivision if X is cell complex
- $\Delta(P)$ is special, min nonfaces all have two elements $\{x,y\}$ x,y incomparable

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posets have topological aspect.

P, P' homotopy equivalent $\Longleftrightarrow$ $\Delta(P), \Delta(P')$ homotopy equivalent.

when simplicial homology of $\Delta(P)$ used in applications and varying methods yield P, P' but same answer, reason is homotopy equivalence

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COUNTING

Given a (finite) poset P, define

zeta ~~matrix~~ function $\zeta(x,y) = \begin{cases} 1, & x \leq y \\ 0 & \text{else} \end{cases}$

think of as incidence matrix for $x \leq y$ in P

$\zeta^{-1} = \mu$ Möbius function of P

(crucial in counting applications)

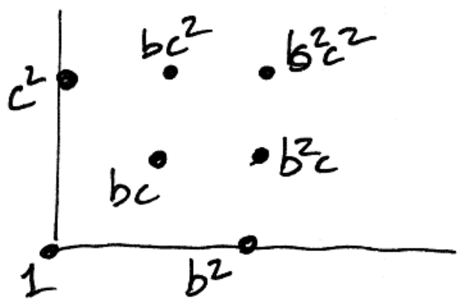
Proposition $\mu(x,y) = \tilde{\chi}(\Delta((x,y)))$

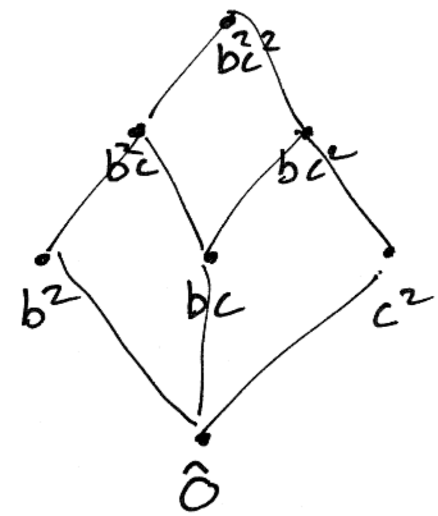
order complex of open interval (x,y) in P

reduced Euler characteristic

example: lcm poset of a monomial ideal

$I = (b^2, bc, c^2)$





	$\hat{0}$	b^2	bc	c^2	b^2c	bc^2	b^2c^2
$\hat{0}$	1	1	1	1	1	1	1
b^2		1			1		1
bc			1		1	1	1
c^2				1		1	1
b^2c					1		1
bc^2						1	1
b^2c^2							1

ζ

	$\hat{0}$	b^2	bc	c^2	b^2c	bc^2	b^2c^2
$\hat{0}$	1	-1	-1	-1	1	1	0
b^2		1			-1		0
bc			1		-1	-1	1
c^2				1		-1	0
b^2c					1		-1
bc^2						1	-1
b^2c^2							1

μ

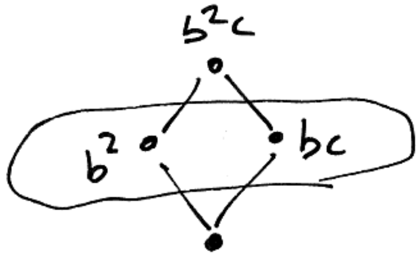
if $x < y$ cover, $c_0 = x < y = c_1$ is chain length 1

$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3$

diagonal entries separate, always 1

rest $x < y$ covers $\Rightarrow \Delta((x,y)) = \{\emptyset\}$

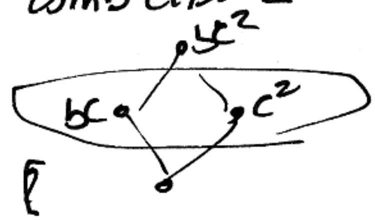
$\tilde{\chi} = -1$



$[\hat{0}, b^2c]$

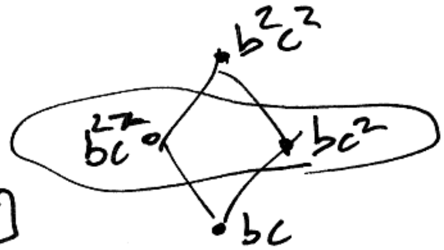
$\tilde{\chi} = 1$

comb class 1



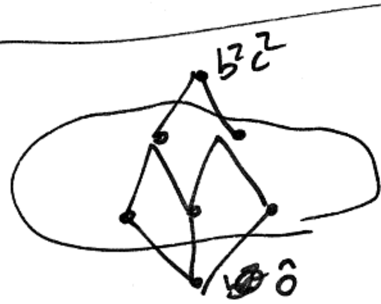
$[\hat{0}, bc^2]$

$\tilde{\chi} = 1$

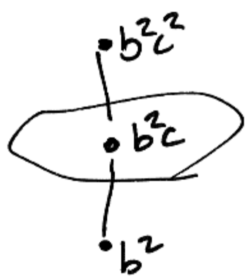


$[bc, b^2c^2]$

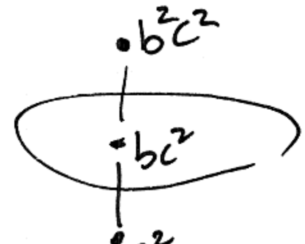
$\tilde{\chi} = 1$



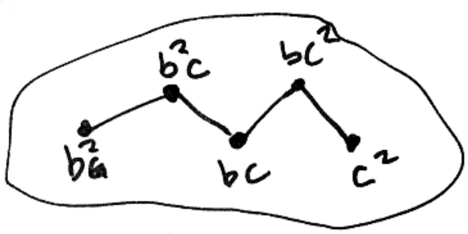
$[\hat{0}, b^2c^2]$



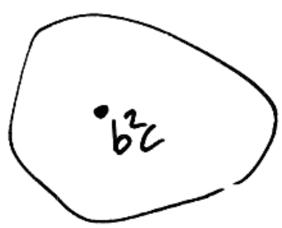
$[b^2, b^2c^2]$



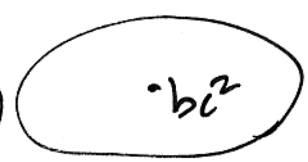
$[c^2, b^2c^2]$



$\tilde{\chi} = 0$



$\tilde{\chi} = 0$



$\tilde{\chi} = 0$

Another example

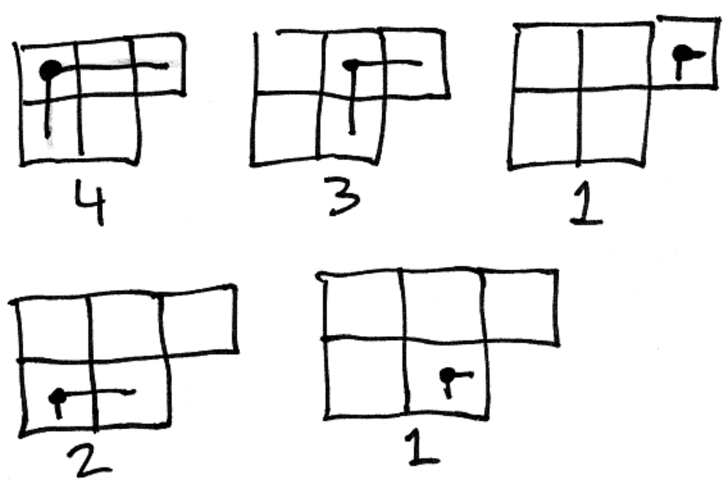
coef in rep thy S_n, $GL(n)$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

$n!/$product of hook lengths

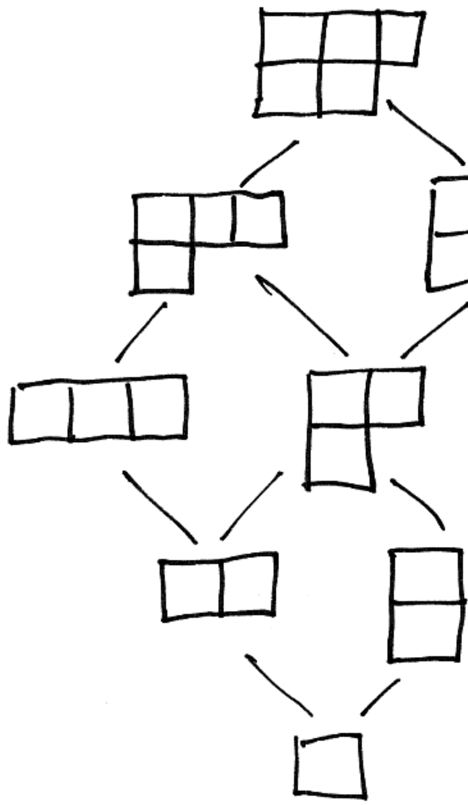


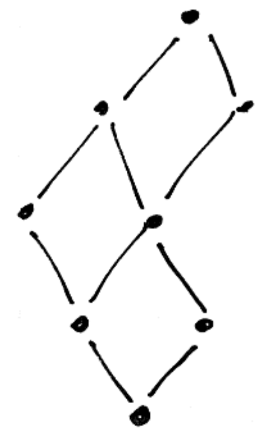
$5!/4\cdot2\cdot3 = 5$ ✓

How to understand proof?

Interpret coef in standard combinatorial way?

(7) comb class 2

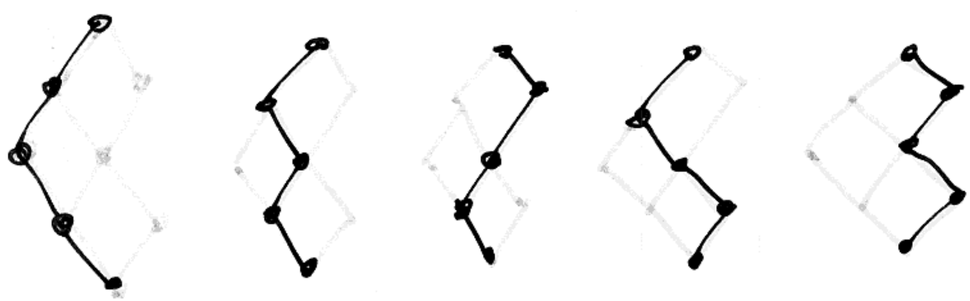




P

$\Delta(P)$

facets of $\Delta(P) =$ coef in ρ rep thy



Now start chapter two, Stanley

2.1 Inclusion-exclusion

$S = \{1..n\}$

$V = \{ f : 2^S \to k \}$

$\phi : V \to V \qquad \phi f(T) = \sum_{Y \supseteq T} f(Y)$

ϕ^{-1} exists, has form $\quad \phi^{-1} f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y)$

proof $\psi : V \to V$ by

$$(\phi\psi f)(T) = \phi\left[\sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y)\right]$$

$$= \sum_{Z \supseteq T} \sum_{Y \supseteq Z} (-1)^{|Y-T|} f(Z)$$

$$= \sum_{Y \supseteq T} (\psi f)(Y) = \sum_{Z \supseteq Y} (-1)^{|Z-}$$

$$\psi(\phi f)(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} (\phi f)(Y)$$

$$= \sum_{Y \supseteq T} (-1)^{|Y-T|} \sum_{Z \subseteq Y} f(Z)$$

$$= \sum_{Z \supseteq T} \underbrace{\left(\sum_{Z \supseteq Y \supseteq T} (-1)^{|Y-T|} \right)}_{\begin{cases} 1, & Z = T \\ 0, & Z \supset T \end{cases}} f(Z)$$

$\left(\begin{array}{l} (1+t)^n \\ t = -1 \Rightarrow 0 \end{array} \right)$

Think of set S ~~what~~ list of properties for external universe of objects

$f(U) =$ # objects w/ exactly properties in U

$\emptyset f(T) = \sum_{U \supseteq T} f(U) =$ # objects w/ at least properties in T

easier to compute latter.

to count objects w/ no properties

$\emptyset f(\emptyset)$ #objects have at least no properties

$-\emptyset f(\{1\}) - \emptyset f(\{2\}) - \cdots - \emptyset f(\{n\})$

subtract out at least propertie $\{i\}$

$+\emptyset f(\{1,2\}) + \cdots$

$- \; + \; .$

"derangement problem"

#($\pi \in S_n$ having no fixed points)

propertie i = has fixed point at i.

$\emptyset f(T) =$ # perms fixed at least at set T

$=$ # perms on $n-|T|$ letters

$= (n-|T|)!$

$$n! - n(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \dots \pm \binom{n}{n} 0!$$

$$= n!\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \dots \pm \tfrac{1}{n!}\right)$$

$= \text{nearest integer to } n!/e$

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example from alg geometry

$I = (\underline{\underline{b^2}} - ac, \underline{\underline{bc}} - ad, \underline{\underline{c^2}} - bd) \subset k[a,b,c,d]$

ideal of twisted cubic curve C

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$
$$(s,t) \longmapsto (s^3, s^2t, st^2, t^3)$$

$f(m) = \dim (S/I)_m$ Hilbert function of C

describes C

combinatorial analogue (same Hilb fn)

$\mathrm{in}(I) = (b^2, bc, c^2) \subset k[a,b,c,d]$

$f(m) = \#$ monoms not contained in $\mathrm{in}(I)$ in degree m

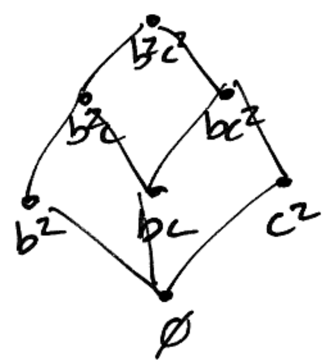
set $S = \{ b^2, bc, c^2 \}$

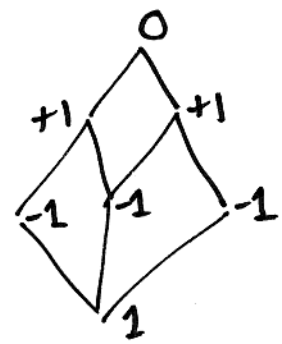
property "monomial divisible by given gen"

(all monoms) − (multiples of b^2)
− (multiples of bc)
− (multiples of c^2)
+ (multiples of ~~b^2c~~ b^2 and bc)
+ (multiples of b^2 and c^2)
+ (" bc and c^2)
− (mult " b^2, bc, c^2)

hierarchy collapses (typical) with cancellation of like terms

<u>poset</u>





$\mu(0,x)$ (row of inverse matrix)

How do we carry out this count <u>generating functions</u>

\# monoms deg m in 1 var $\sum_{m=0}^{\infty} t^m = \frac{1}{1-t}$

\# monoms in vars, a, b

$$(1+a+a^2+\ldots)(1+b+b^2+\ldots)$$
$$= (1+a+b+a^2+ab+b^2+a^3+\ldots)$$
$$\Downarrow a=b=1$$
$$(1+2t+3t^2+\ldots)$$

$$\left(\frac{1}{1-t}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}$$

so generating function for

$\mathbb{P}^r = \operatorname{Proj} k[x_0, \ldots, x_r]$

is $\frac{1}{(1-t)^{r+1}} = \sum_{m=0}^{\infty} \binom{r+m}{r} t^r$

bars and stars check

$\underline{*} \mid \underline{*} \; \underline{*} \mid \iff ab^2$

need slots for m vars

r dividers

choose location of r dividers

How to shift count? multiples of b^2c in $k[a,b,c,d]$?

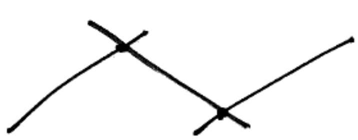
$$\frac{t^3}{(1-t)^4}$$

$$\cancel{\frac{1}{(1-t)^4} - \frac{t^2}{(1-t)^4} - \frac{t^2}{(1-t)^4}}$$

$$\frac{1-3t^2+2t^3}{(1-t)^4}$$

$$= \frac{3}{(1-t)^2} - \frac{2}{(1-t)}$$

$$= 3\mathbb{P}^1 - 2\mathbb{P}^0$$



3 lines, 2 points overlap

stick twisted cubic