Math Prize Exam

Columbia University and Barnard College, March 26, 2009

Instructions:

Please have fun. You may work any problems that you like. The problems are in no particular order, and vary in difficulty. They are likely to be correctly stated, but do not assume this. Do not ask for clarifications.

Each problem is worth 10 points. Credit will only be given for substantial progress. Correct answers are worth 8 to 10 points at the discretion of each grader, with perfect scores reserved for solutions that are correct, concise, and clear. No credit will be given for effort, and we will not reward lengthy answers. If you find yourself starting to write a novel, ask yourself if you would be better off solving several other problems. It is up to you to use your time wisely.

This is a contest, and following these rules correctly is part of the contest:

Please complete the cover page, identifying yourself and your school, so that we may associate you with each problem that you submit. Note the two-digit exam number on your cover page; you will use this exam number for each problem that you submit.

Each problem will be submitted separately, and graded separately. Submit all work on the provided forms; scrap paper will be discarded. Do not work more than one problem per form; you may use both sides of each form. Label the front of each form with your two-digit exam number, the problem number, and a consecutive page number *for that problem*. In other words, each form has four boxes to fill out; fill out all four boxes correctly on the front of every form that you submit. If you need to use more than one form for a given problem, staple the forms together *for that problem*. If you staple more than one problem together, only the first problem will be graded.

[**1**] Construct a function $f : \mathbb{R} \to \mathbb{R}$ whose graph is dense in $\mathbb{R}^2$. (I.e. for any $(x, y) \in \mathbb{R}^2$ there exists a sequence $x_n \to x$ with $f(x_n) \to y$).

[**2**] Let $y_1, .., y_n$ and $z_1, .., z_n$ be points in $\mathbb{R}^2$ (or $\mathbb{R}^m$) with different centers of mass, i.e. $\sum y_i/n \neq \sum z_i/n$. Show that there exists (an infinity of) x such that

$$\sum |x - y_i| = \sum |x - z_i|$$

[**3**] Show that a disk of area 1 can be covered by a given collection of disks with total area 25.

[**4**] Show that solutions of

$$f''(t) = 2\sin(t^3 + f(t))$$

are asymptotic to lines as $t \to \infty$. Is the result still true if we replace t^3 by t^2?

[**5**] How many positive integers $\leqslant 600$ are the product of three distinct primes?

[**6**] Consider a rectangle with side lengths A, B. Suppose that there is a subdivision of this rectangle into a finite number of smaller rectangles such that each smaller rectangle has one side whose length is an integer. Show that A or B is an integer.

[**7**] Let $A = m^2 + mn - n^2$, with $m, n \in \mathbb{Z}$. Show $A \neq +3$, and $A \neq -3$.

[**8**] Let B be the $n \times n$ *chess board* consisting of the *squares*

$$\{\ (x, y)\ \mid\ x, y \in \mathbb{Z},\ 1 \leqslant x \leqslant n,\ 1 \leqslant y \leqslant n\ \}$$

A path $(x_1, y_1), \ldots, (x_k, y_k)$ in B consists of *knight's moves* if the length of each step is $\sqrt{5}$. Does there exist a path which visits all but one square exactly once?

[9] Show that for every positive integer s there is a positive integer t such that the decimal expansion of their product st consists only of the digits 0 and 7.

[10] Show that for every positive integer n the integer $2^n + 3^n$ is not the square of an integer.

[11] Let p be a prime number, all of whose digits are 1's. Show that the number of digits of p must be prime.

[12] Prove that $(\sqrt{5}+2)^{1/3} - (\sqrt{5}-2)^{1/3} = 1.$

[13] Show that if n is a natural number then $n^4 + 4$ is a prime number only if $n = 1$.

[14] Let x and y be positive numbers with $x + y = 6$. Show that $xy \leqslant 9$ and $xy^2 \leqslant 32$. In each case, for what x and y does one have equality?

[15] Find all positive integers $k_1, \ldots, k_m, \ell, n$ with $\ell \leqslant n$ and $m \geqslant 2$ for which both

$$k_1 + \ldots + k_m = \ell$$

and

$$\binom{n-1}{k_1-1} + \ldots + \binom{n-1}{k_m-1} = \binom{n-1}{\ell-1}$$

Here $\binom{n}{k}$ is the usual binomial coefficient.

[16] Use Euler's formula

$$\int_0^\infty v^n e^{-v}\, dv = n! \quad \text{for} \quad n = 0, 1, 2, \ldots$$

to show that

$$\int_0^1 \frac{dx}{x^x} = \sum_{n=1}^\infty \frac{1}{n^n}$$