Columbia-Barnard
MATHEMATICS PRIZE EXAM

April 17, 2008

Please print your name:

Email:

Indicate school:

[ ] First-year [ ] Sophomore [ ] Junior [ ] Senior

Expected date of graduation:

This is a three-hour exam. No calculators. Please print your name on each booklet that you hand in. It is not expected that anyone will complete the entire exam. Submit your paper even if you have done no more than one or two problems. Submit also partial answers since credit will be given for relevant progress made on a problem.

Problems

1. A standard deck of 52 playing cards is randomly shuffled. The top six cards are discarded unseen, and the remaining cards are revealed one by one. What is the probability that all six red face cards (jack, queen, king of hearts or diamonds) appear before any black face card appears?

2. Find a quadratic polynomial f with real coefficients satisfying

$$|f(0)| = 1,\ |f(1)| = 1,\ |f(2)| = 1,\ f(4) = 5.$$

3. Let I be the $n \times n$ identity matrix. Find a matrix A with real entries whose square is $-I$, or prove that this is impossible.

4. Fermat proved that the number $2^{37} - 1 = 137438953471$ was composite by finding a prime factor p in the range $200 < p < 300$. What is p?

5. Let $p(x)$ be a polynomial in a single variable x with real coefficients,

$$p(x) = \sum_{i=0}^{m} a_i x^i.$$

Suppose moreover that for any integer n, $p(n)$ is also an integer. Give an example of a polynomial p of the above form whose coefficients a_i are not all integers. What can you say about the coefficients a_i of all polynomials of this type?

6. Find all triples of positive integers p, q, r for which, for each positive integer n, the p^{th} power of the sum of the q^{th} powers of the first n positive integers equals the sum of the r^{th} powers of the first n positive integers.

7. You have an inexhaustible supply of 37 cent stamps, 39 cent stamps, and "Forever Stamps," which are always worth the current First Class one ounce rate (currently 41 cents). What is the largest value that you cannot compose by some combination of these stamps? What happens as the First Class rate goes up?

8. A circular wire of unit radius is taken from a water bath with a linearly varying temperature gradient and immediately placed in an insulating fluid. How does the temperature distribution evolve over time?

9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(xyz) \;=\; yz\, f(x) + xz\, f(y) + xy\, f(z)$$

for all x, y, z.

10. Evaluate

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \dots}}}}}$$

Prove that this expression converges to your answer.