Columbia-Barnard
MATHEMATICS PRIZE EXAM

March 22, 2007

Please print your name:

Email:

Indicate school:

[ ] First-year [ ] Sophomore [ ] Junior [ ] Senior

Expected date of graduation:

This is a three-hour exam. No calculators. Please print your name on each booklet that you hand in. It is not expected that anyone will complete the entire exam. Submit your paper even if you have done no more than one or two problems. Submit also partial answers since credit will be given for relevant progress made on a problem.

Problems

1. Evaluate
$$\int_{0\le y\le x} e^{-x^2-y^2}\,dxdy.$$

2. Suppose that we have 5 points on the unit 2-sphere $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}$. Show that there is a closed half-sphere (one half of the sphere bounded by a great circle) that contains at least 4 of these points.

3. Let $f:\mathbb{R}\to\mathbb{R}$ ($\mathbb{R}$ = real number) and suppose $f(0)=0$, $f(1)=1$ and $f(x+y)=f(x)+f(y)$ for all x,y. Must $f(x)=x$ for all x?

4. Show that 31,275,407 cannot be written as the sum of three perfect squares.

5. If 2^n-1 is prime, show n is prime.

6. If A is a 3×2 matrix and B is a 2×3 matrix, then AB is not equal to I.

7. Let $p(x)$ be a polynomial with integer coefficients. Let $A=(a_1,a_2)$, $B=(b_1,b_2)$ be two points (where a_1,a_2,b_1,b_2 are integers) that lie on the curve $y=p(x)$. Assume that the distance between A and B is an integer. Show that the line passing through A and B is parallel to the x-axis.

8. Which number is larger, e^π or π^e?

9. Let M_n be the set of $n\times n$ real matrices. For $n>m$, show that there is no bijection $f:M_n\to M_m$ such that $f(X)f(Y)=f(XY)$ for all $X,Y\in M_n$.

10. The ellipse $9x^2+y^2=9$ has the following two parametrizations by an angle in $[0,2\pi)$: The first is by the polar coordinate θ of the point (x,y), and the second is by the angle ϕ satisfying $x=\cos(\phi)$, $y=3\sin(\phi)$. Find the maximum value of the difference $\theta-\phi$.

11. Show that the only positive integer solutions of $x^y=y^x$ with $x\neq y$ are $(x,y)=(2,4)$ and $(4,2)$.

12. Let $F=\mathbb{C}(x)$ denote the field of rational functions over the complex numbers in one variable x. Find all the automorphisms of F which fix $\mathbb{C}$