Columbia-Barnard
MATHEMATICS PRIZE EXAM

March 30, 2006

Please print your name:

Email:

Indicate school:

[ ] First-year   [ ] Sophomore   [ ] Junior   [ ] Senior

Expected date of graduation:

This is a three-hour exam. Please print your name on each booklet that you hand in. Submit your paper even if you have done no more than one or two problems. It is not expected that anyone will complete the entire exam. However, work as many problems as you can because partial credit will be given for significant progress made on a problem.

Problems

1. Two old women started a trip at sunrise and each walked at constant velocity. One went directly from village A to village B and the other from village B to village A. They met at noon and, continuing with no stop, arrived respectively at villages B and A at 4PM and 9PM. At what time was sunrise on that day?

2. Show that if x, y, z satisfy the equations

$$x + 1/x = y + 1/y = z + 1/z$$

then one of the following equations holds: $x = y, \quad x = z, \quad y = x\,.$

3. (a). If three planes in Euclidean 3-space meet in a single point, show that a plane that is not parallel to any of them and does not pass through their common point encloses a bounded region with them.
(b) What is the maximum number of planes you can arrange in Euclidean 3-space which enclose exactly one bounded region? Prove your answer

4. Suppose $r \leq n \leq m$. Show that an $m \times n$ matrix has rank $\leq r$ if and only if it can be written as the sum of r rank 1 matrices.

5. A cylindrical pot is 12 inches diameter and 12 inches deep and is filled to the brim with icecream. The only available implement for removing the icecream is a rigid square of thin metal 15 inches square. What volume of icecream can be removed with this implement?

6. Can a continuous function on the real line have the property that $f(x+1)$ is rational if and only if $f(x)$ is irrational. Prove your answer.

7. Prove that if A is an $n \times n$ real matrix and $A^T A = A^2$, then $A^T = A$.

8. Explain how how you would compute a good approximation to $\sqrt{2}$ *by hand* without a calculator. (Or just do it.)

9. Compute $\int_0^1 \int_0^{\sqrt{\pi^2 - x^2}} \sin(x^2 + y^2)\, dy\, dx$

10. Does the improper integral $\int_1^\infty \frac{\sin x}{x}\, dx$ converge? Prove your answer.

11. You want to distribute 1000 \$1 dollar bills among a collection of sealed envelopes so that whatever whole number of dollars from 1 to 1000 you are asked for, you can provide the exact amount by handing over some collection of the envelopes. What is the least number of envelopes you need? Prove your answer.