Columbia-Barnard
MATHEMATICS PRIZE EXAM

April 1, 2004

Please print your name:

Email address:

Indicate school:

[ ] First-year [ ] Sophomore [ ] Junior [ ] Senior

Expected date of graduation:

This is a three-hour exam. It has 15 questions. It is not expected that anyone will complete the entire exam. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.
Please print your name on each booklet that you hand in.

Columbia/Barnard Mathematics Prize Exam
Thursday, April 1, 2004

1. A compulsive arithmetician adds up the sequence of numbers $1, 2, \ldots, n$ for a certain value of n. Unfortunately he misses one number in the sequence. Assuming he made no other errors, and got the answer 2004, find which number he missed.

2. Show that the equation $a^2 + 5ab + 7b^2 = 8$ has no integral solutions.

3. Find the smallest constant c such that $x^3+y^3 \leq c(x+y)^3$ for all $x, y \geq 0$

4. Let $f(n) = 1! + 2! + 3! + \ldots + n!$. Show that for every natural number k there exist polynomials $p_k(x)$ such that

$$f(n+k) \leq p_k(n) \cdot f(n) \quad \text{for all } n\,.$$

For each k, find the smallest degree of a polynomial $p_k(x)$ that satisfies the above inequality.

5. How many continuous functions $f \colon [0,1] \to [0, \infty)$ are there which satisfy the following three constraints:

$$\int_0^\infty f(x)dx = 1 \quad \int_0^\infty xf(x)dx = 2 \quad \int_0^\infty x^2 f(x)dx = 3$$

6. A clockmaker accidentally puts two identical hands on a clock (so the minute and hour hands of the clock are indistinguishable). The clock works fine otherwise. How many times during a 12 hour period does the clock display an ambiguous time?

7. Let $\mathbb{N}_n$ denote the set $\{1, 2, 3, \ldots, n\}$. How many subsets S are there of $\mathbb{N}_n$ which have exactly k elements and satisfy

$$|i - j| \geq 3 \quad \text{for all } i, j \in S\ ?$$

8. Let a_n be a sequence of real numbers such that the series

$$\sum a_n + 3a_{n+1}$$

converges. Show that $\sum a_n$ also converges.

9. Find the probability that January 1st of a randomly picked year is a Sunday. Recall that a year n is a leap year if either n is divisible by 4 but not by 100 or n is divisible by 400.

10. Let $x, y \in \mathbb{R}$ and for an integer $n \geq 2$ let $A = [a_{ij}]$ be the following $n \times n$ matrix:

$$a_{ij} = \begin{cases} x^{j-i} & i < j \\ 1 & i = j \\ y^{i-j} & i > j \end{cases}$$

For example, if $n = 4$ then A looks like

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ y & 1 & x & x^2 \\ y^2 & y & 1 & x \\ y^3 & y^2 & y & 1 \end{bmatrix}$$

Find the determinant of A.

11. A circle with a marked point on it rolls along a straight horizontal line. Find the length of the curve traced out by the marked point as the circle completes one full revolution. The radius of the circle is 1.

12. Find all positive integers n for which

$$n^{n+2} = (n+2)^n$$

13. A railway line travels in a loop around a city. The rails are 5 feet apart. How much longer is the outer rail than the inner one. (You should not assume that the route is circular, but you may assume that the rails follow a smooth curve in a plane.)

14. Two infinite straight lines meet at an angle θ. A straight segment of length 1 moves with one endpoint on one line and one endpoint on the other line. Show that the midpoint of the segment moves along an ellipse and compute, in terms of θ, the long and short axes of this ellipse.

15. Three points move in the plane. Each point moves at constant speed along a straight line. Let S be the set of times $t \in \mathbb{R}$ that the points are lined up (i.e., there is a straight line passing through all three points). What are the possible values for the size of S?

End of Exam