Solutions to 2003 Prize Exam

1. Show that $i^i$ is a real number (where $i = \sqrt{-1}$). Which real number is it?
   **Answer:** $i = e^{\pi i/2}$ so $i^i = e^{\pi i^2/2} = e^{-\pi/2}$

2. Let $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Prove the following for $n \geq 1$ ($\lfloor x \rfloor$ is “integer part”: greatest integer not exceeding $x$)
   \[2^{n-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\]
   **Answer:** Several solutions were offered. 1. By the binomial theorem, $(1+x)^n = \sum x^k \binom{n}{k}$. Substitute $x = 1$ and $x = -1$ and add the two resulting equations and the desired result drops out.
   2. Binomial coefficients satisfy a recursion $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Substitute this with $K = 2k$ into the formula to be proved and you get the binomial expansion of $(1+1)^{n-1}$.

3. Let $A$ be a symmetric $n \times n$ real matrix. Prove that $A$ can be written in the form $B^tB$ for some real $n \times n$ matrix $B$ if and only if the eigenvalues of $A$ are nonnegative. ($B^t$ denotes the transpose of $B$.)
   **Answer:** Only if: If $\lambda$ is an eigenvalue of $B^tB$ with eigenvector $v$ then $0 \leq \langle Bv, Bv \rangle = v^tB^tBv = \lambda v^tv$ and $v^tv > 0$ so $\lambda \geq 0$.
   If: A symmetric matrix is diagonalizable (Gram-Schmidt) so there exists invertible $P$ with $P^tAP = \text{diag}(\lambda_1, \ldots, \lambda_n) = D^tD$ with $D = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ so $P^tAP = D^tD$, so $A = (P^t)^{-1}D^tDP^{-1} = B^tB$ with $B = DP^{-1}$.

4. Let $N$ be a 6-digit number, the digits being distinct and in the set $1, 2, 3, 4, 5, 6, 7, 8, 9$ (so that 0 does not occur). Assume that the numbers $2N$, $3N$, $4N$, $5N$, $6N$ are all 6-digit numbers and that each is a permutation of the digits in $N$. Find $N$.
   **Answer:** 142857 works. It is the only solution. Proof: Let $N = a_1a_2a_3a_4a_5a_6$ and $S = \{a_1, \ldots, a_6\}$. Then $a_1 = 1$ since otherwise $6N$ is too large. Thus $1 \in S$. Next, $a_6 = 7$ since otherwise the last digits of $N, 2N, \ldots, 6N$ either include 0 (for $a_6 = 2, 4, 5, 6, 8$) or form a six-element set not containing 1 (for $a_6 = 3, 9$). Thus $S = \{7, 4, 1, 8, 5, 2\}$. Since $6N$ must start with 8 we can rule out $a_2 = 2$ ($6N$ too small) or $a_2 \geq 5$ ($6N$ too large) so $a_2 = 4$. Now $a_5 \neq 2$ since if $N = .27$ then $4N = .08$ and $a_5 \neq 8$ since if $N = .87$ then $3N = .67$. Thus $a_5 = 5$, so $N = 142857$ or $148257$. Checking $2N$ shows that 148257 does not work.

5. Consider the function $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. Show that this is a continuous function of $s$. Prove that

$$\zeta(s) = \prod_{p=\text{prime}} \frac{1}{1-1/p^s}$$

Hint: Recall that each natural number can be uniquely decomposed into its prime factors.
What is $\zeta(1)$? What does this say about how many prime numbers there are?

**Answer:** The series converges for $s > 1$ (“p-series”). Continuity on any interval $[a, b]$ with $1 < a < b$ follows from uniform convergence of the series on this interval, so continuity on $(1, \infty)$ follows. Expand $1/(1 - 1/p^n)$ as $\sum_n (1/p^n)^n$. Let $p_1, \ldots, p_k$ be the first $k$ primes. Multiplying out $(\sum_n (1/p^n)^n) \cdots (\sum (1/p^n)^n)$ gives the sum of all terms of the form $1/(p_1^{n_1} \cdots p_k^{n_k})^n$, i.e., the sum of all $1/n^s$ for which the only prime factors of $n$ are in the first $k$ primes. Thus the limit as $k \to \infty$ proves the desired product formula, and the fact that the series for $\zeta(1)$ diverges shows the product is infinite, i.e., there are infinitely many primes.

6. Show that for odd $n > 1$, $\phi_{2n}(x) = \phi_n(-x)$, where $\phi_n$ is the $n$th cyclotomic polynomial (polynomial of minimal degree whose roots are the primitive $n$-th roots of 1).

**Answer:** $\alpha$ is a primitive $2n$-th root if $\alpha^{2n} = 1$ and $\alpha^d \neq 1$ for any proper divisor $d$ of $2n$. Then $\alpha^n = -1$, (since its square is 1) so $(-\alpha)^n = (-1)^n(-1) = 1$, so $-\alpha$ is an $n$-th root of 1. It is a primitive $n$-th root, since $(-\alpha)^d = 1$ for $d$ a proper divisor of $n$ would imply $\alpha^{2d} = 1$. Similarly one shows that if $\alpha$ is a primitive $n$-th root of 1 then $-\alpha$ is a primitive $2n$-th root. It follows that $\phi_{2n}(x)$ and $\phi_n(-x)$ both have the same roots, so they are equal up to sign. The sign is $(-1)^{deg\phi_n}$ which is +1 since there are an even number of primitive $n$-th roots if $n \neq 2$ (if $\alpha$ is a primitive $n$-th root then so is $\alpha^{-1}$).

7. If $z$ is a complex number prove that $(\max(Re(z^n), Im(z^n)))^{1/n}$ converges to $|z|$ as $n \to \infty$.

**Answer:** As most participants noticed, this should have read $\max(|Re(z^n)|, |Im(z^n)|)$, since otherwise the claim is false for most $z$. For any complex $w$ one has $|w|^2 = |Re(w)|^2 + |Im(w)|^2$, so at least one of $|Re(w)|$ and $|Im(w)|$ exceeds $\sqrt{\frac{2}{2}} |w|$. Applied to $z^n$ this gives

$$\frac{\sqrt{2}}{2} |z^n| \leq \max(|Re(z^n)|, |Im(z^n)|) \leq |z^n|$$

so

$$\left(\frac{\sqrt{2}}{2}\right)^\frac{1}{n} |z| \leq \max(|Re(z^n)|, |Im(z^n)|)^\frac{1}{n} \leq |z|.$$

Taking limit as $n \to \infty$ gives the result since $(\frac{\sqrt{2}}{2})^{1/n} \to 1$.

8. Show that if $b^2 - 4ac$ is negative then the graph of $ax^2 + bxy + cy^2 = 1$ represents an ellipse that encloses an area of $2\pi/\sqrt{4ac - b^2}$.

**Answer:** Completing the square gives the equation $(ax + \frac{b}{2a}y)^2 + (\frac{4ac-b^2}{4a})y^2 = 1$. The linear transformation $u = ax + \frac{b}{2a}y, v = \sqrt{\frac{4ac-b^2}{4a}}y$ converts this to a circle of radius 1. By change of coordinates, the area is thus $\int \int_A |\frac{\partial(x, y)}{\partial(u, v)}| \, du \, dv$, integrated over the unit disk in $(u, v)$-coordinates. Since

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| \cdot \left|\frac{\partial(u, v)}{\partial(x, y)}\right|^{-1} = \left|\begin{array}{c} a \\ \sqrt{\frac{4ac-b^2}{4a}} \end{array}\right|^{-1} = \left(\frac{\sqrt{4ac-b^2}}{2}\right)^{-1},$$
this gives the desired answer.

9. Consider the sequence $a_1 = 3, a_{n+1} = a_n + \sin(a_n)$. Show that this sequence converges to $\pi$.

**Answer:** Several correct solutions were offered. One was to observe that $a_n$ is an increasing sequence bounded above by $\pi$ so it has a limit $l$ with $0 < l \leq \pi$. Once one knows it has a limit $l$, taking limit of the defining equation $a_{n+1} = a_n + \sin(a_n)$ gives $l = l + \sin(l)$, from which $l = \pi$ follows.

Here is one using the Taylor series for $\sin(x)$ that gives the rate of convergence. Let $x$ be the “error” in approximation of $a_n$ to $\pi$, so $a_n = \pi - x$. Note that $\sin(\pi - x) = \sin x$. Thus $\sin(a_n) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, so

$$a_{n+1} = a_n + \sin a_n = \pi - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

which differs from $\pi$ by less than $\frac{x^3}{6}$. Thus the error decreases at better than cubic rate, i.e., if $a_n$ approximates $\pi$ to $k$ digits, then $a_{n+1}$ will approximate to better than $3k$ digits.