Solutions to 2003 Prize Exam

1. Show that i^i is a real number (where $i = \sqrt{-1}$). Which real number is it?
Answer: $i = e^{\pi i/2}$ so $i^i = e^{\pi i^2/2} = e^{-\pi/2}$

2. Let $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Prove the following for $n \geq 1$ ($\lfloor x \rfloor$ is "integer part": greatest integer not exceeding x)

$$2^{n-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}$$

Answer: Several solutions were offered. 1. By the binomial theorem, $(1+x)^n = \sum x^k \binom{n}{k}$. Substitute $x = 1$ and $x = -1$ and add the two resulting equations and the desired result drops out.
2. Binomial coefficients satisfy a recursion $\binom{n}{K} = \binom{n-1}{K-1} + \binom{n-1}{K}$. Substitute this with $K = 2k$ into the formula to be proved and you get the binomial expansion of $(1+1)^{n-1}$.

3. Let A be a symmetric $n \times n$ real matrix. Prove that A can be written in the form $B^t B$ for some real $n \times n$ matrix B if and only if the eigenvalues of A are nonnegative. (B^t denotes the transpose of B.)
Answer: Only if: If λ is an eigenvalue of $B^t B$ with eigenvector v then $0 \leq (Bv)^t Bv = v^t B^t Bv = \lambda v^t v$ and $v^t v > 0$ so $\lambda \geq 0$.
If: A symmetric matrix is diagonalizable (Gram-Schmidt) so there exists invertible P with $P^t AP = diag(\lambda_1, \ldots, \lambda_n)$. If each λ_i is non-negative then $diag(\lambda_1 \ldots, \lambda_n) = D^t D$ with $D = diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ so $P^t AP = D^t D$, so $A = (P^t)^{-1} D^t DP^{-1} = B^t B$ with $B = DP^{-1}$.

4. Let N be a 6-digit number, the digits being distinct and in the set 1,2,3,4,5,6,7,8,9 (so that 0 does not occur). Assume that the numbers 2N, 3N, 4N, 5N, 6N are all 6-digit numbers and that each is a permutation of the digits in N. Find N.
Answer: 142857 works. It is the only solution. Proof: Let $N = a_1 a_2 a_3 a_4 a_5 a_6$ and $S = \{a_1, \ldots, a_6\}$. Then $a_1 = 1$ since otherwise $6N$ is too large. Thus $1 \in S$. Next, $a_6 = 7$ since otherwise the last digits of $N, 2N, \ldots, 6N$ either include 0 (for $a_6 = 2, 4, 5, 6, 8$) or form a six-element set not containing 1 (for $a_6 = 3, 9$). Thus $S = \{7, 4, 1, 8, 5, 2\}$. Since $6N$ must start with 8 we can rule out $a_2 = 2$ ($6N$ too small) or $a_2 \geq 5$ ($6N$ too large) so $a_2 = 4$. Now $a_5 \neq 2$ since if $N = ..27$ then $4N = ..08$ and $a_5 \neq 8$ since if $N = ..87$ then $3N = ..67$. Thus $a_5 = 5$, so $N = 142857$ or 148257. Checking $2N$ shows that 148257 does not work.

5. Consider the function $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. Show that this is a continuous function of s. Prove that

$$\zeta(s) = \prod_{p=prime}^{\infty} \frac{1}{1 - 1/p^s}$$

Hint: Recall that each natural number can be uniquely decomposed into its prime factors.

What is $\zeta(1)$? What does this say about how many prime numbers there are?
Answer: The series converges for $s > 1$ ("p-series"). Continuity on any interval $[a, b]$ with $1 < a < b$ follows from uniform convergence of the series on this interval, so continuity on $(1, \infty)$ follows. Expand $1/(1 - 1/p^s)$ as $\sum_n (1/p^s)^n$. Let $p_1, \ldots, p_k$ be the first k primes. Multiplying out $(\sum_n (1/p_1^s)^n) \ldots (\sum (1/p_k^s)^n)$ gives the sum of all terms of the form $1/(p_1^{n_1} \ldots p_k^{n_k})^s$, i.e., the sum of all $1/n^s$ for which the only prime factors of n are in the first k primes. Thus the limit as $k \to \infty$ proves the desired product formula, and the fact that the series for $\zeta(1)$ diverges shows the product *is* infinite, i.e., there are infinitely many primes.

6. Show that for odd $n > 1$, $\phi_{2n}(x) = \phi_n(-x)$, where ϕ_n is the nth cyclotomic polynomial (polynomial of minimal degree whose roots are the primitive n-th roots of 1).
Answer: α is a primitive $2n$-th root if $\alpha^{2n} = 1$ and $\alpha^d \neq 1$ for any proper divisor d of $2n$. Then $\alpha^n = -1$, (since its square is 1) so $(-\alpha)^n = (-1)^n(-1) = 1$, so $-\alpha$ is an n-th root of 1. It is a primitive n-th root, since $(-\alpha)^d = 1$ for d a proper divisor of n would imply $\alpha^{2d} = 1$. Similarly one shows that if α is a primitive n-th root of 1 then $-\alpha$ is a primitive $2n$-th root. It follows that $\phi_{2n}(x)$ and $\phi_n(-x)$ both have the same roots, so they are equal up to sign. The sign is $(-1)^{deg\phi_n}$ which is $+1$ since there are an even number of primitive n-th roots if $n \neq 2$ (if α is a primitive n-th root then so is α^{-1}).

7. If z is a complex number prove that $(max(Re(z^n), Im(z^n)))^{1/n}$ converges to $|z|$ as $n \to \infty$.
Answer: As most participants noticed, this should have read $max(|Re(z^n)|, |Im(z^n)|)$, since otherwise the claim is false for most z. For any complex w one has $|w|^2 = |Re(w)|^2 + |Im(w)|^2$, so at least one of $|Re(w)|$ and $|Im(w)|$ exceeds $\frac{\sqrt{(2)}}{2}|w|$. Applied to z^n this gives

$$\frac{\sqrt{2}}{2}|z^n| \leq max(|Re(z^n)|, |Im(z^n)|) \leq |z^n|$$

so

$$\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{n}} |z| \leq max(|Re(z^n)|, |Im(z^n)|)^{\frac{1}{n}} \leq |z|.$$

Taking limit as $n \to \infty$ gives the result since $(\frac{\sqrt{2}}{2})^{1/n} \to 1$.

8. Show that if $b^2 - 4ac$ is negative then the graph of $ax^2 + bxy + cy^2 = 1$ represents an ellipse that encloses an area of $2\pi/\sqrt{4ac - b^2}$.
Answer: Completing the square gives the equation $(ax + \frac{b}{2a}y)^2 + (\frac{4ac-b^2}{4a})y^2 = 1$. The linear transformation $u = ax + \frac{b}{2a}y, v = \sqrt{\frac{4ac-b^2}{4a}}y$ converts this to a circle of radius 1. By change of coordinates, the area is thus $\int\int_A \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$, integrated over the unit disk in (u, v)-coordinates. Since

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\frac{\partial(u, v)}{\partial(x, y)}\right|^{-1} = \begin{vmatrix} a & b/2a \\ 0 & \sqrt{(4ac-b^2)/4a} \end{vmatrix}^{-1} = \left(\frac{\sqrt{4ac - b^2}}{2}\right)^{-1},$$

this gives the desired answer.

9. Consider the sequence $a_1 = 3$, $a_{n+1} = a_n + \sin(a_n)$. Show that this sequence converges to π.

Answer: Several correct solutions were offered. One was to observe that a_n is an increasing sequence bounded above by π so it has a limit l with $0 < l \leq \pi$. Once one knows it has a limit l, taking limit of the defining equation $a_{n+1} = a_n + \sin(a_n)$ gives $l = l + \sin(l)$, from which $l = \pi$ follows.

Here is one using the Taylor series for $\sin(x)$ that gives the rate of convergence. Let x be the "error" in approximation of a_n to π, so $a_n = \pi - x$. Note that $\sin(\pi - x) = \sin x$. Thus $\sin(a_n) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, so

$$a_{n+1} = a_n + \sin a_n = \pi - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

which differs from π by less than $x^3/6$. Thus the error decreases at better than cubic rate, i.e., if a_n approximates π to k digits, then a_{n+1} will approximate to better than $3k$ digits.