Columbia-Barnard
MATHEMATICS PRIZE EXAM

April 2, 2003

Please print your name:

Email:

Indicate school:

[ ] First-year [ ] Sophomore [ ] Junior [ ] Senior

Expected date of graduation:

This is a three-hour exam. Please print your name on each booklet that you hand in. Submit your paper even if you have done no more than one or two problems. It is not expected that anyone will complete the entire exam. However, work as many problems as you can because partial credit will be given for significant progress made on a problem.

Problems

1. Show that i^i is a real number (where $i = \sqrt{-1}$). Which real number is it?

2. Let $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Prove the following for $n \geq 1$ ($\lfloor x \rfloor$ is "integer part": greatest integer not exceeding x)

$$2^{n-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}$$

3. Let A be a symmetric $n \times n$ real matrix. Prove that A can be written in the form $B^t B$ for some real $n \times n$ matrix B if and only if the eigenvalues of A are nonnegative. (B^t denotes the transpose of B.)

4. Let N be a 6-digit number, the digits being distinct and in the set 1,2,3,4,5,6,7,8,9 (so that 0 does not occur). Assume that the numbers 2N, 3N, 4N, 5N, 6N are all 6-digit numbers and that each is a permutation of the digits in N. Find N.

5. Consider the function $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$ for $s > 1$. Show that this is a continuous function of s. Prove that

$$\zeta(s) = \prod_{p=prime}^{\infty} \frac{1}{1 - 1/p^s}$$

Hint: Recall that each natural number can be uniquely decomposed into its prime factors. What is $\zeta(1)$? What does this say about how many prime numbers there are?

6. Show that for odd $n > 1$, $\phi_{2n}(x) = \phi_n(-x)$, where ϕ_n is the nth cyclotomic polynomial (polynomial of minimal degree whose roots are the primitive n-th roots of 1).

7. If z is a complex number prove that $(max(Re(z^n), Im(z^n)))^{1/n}$ converges to $|z|$ as $n \to \infty$.

8. Show that if $b^2 - 4ac$ is negative then the graph of $ax^2 + bxy + cy^2 = 1$ represents an ellipse that encloses an area of $2\pi/\sqrt{4ac - b^2}$.

9. Consider the sequence $a_1 = 3$, $a_{n+1} = a_n + \sin(a_n)$. Show that this sequence converges to π.