Columbia-Barnard

MATHEMATICS PRIZE EXAM

April 5, 2001

Solutions

1. Anne, Barbara, and Carol are the only contestants in a race. Anne started last and during the race she swapped positions with other contestants seven times, ending the race ahead of Barbara. Who won? (Prove your answer).

Anne started in 3rd place. After an odd number of swaps, her position must be even, so after 7 swaps Anne was in 2nd place. Barbara must have been in 3rd place to be behind Anne, so Carol won.

2. Show that $(1-x^a)^{\frac{1}{a}} < (1-x^b)^{\frac{1}{b}}$ for all $x \in (0,1)$ if $0 < a < b$.

Since $x < 1$, $x^a > x^b$, so $1-x^a < 1-x^b$. Then $(1-x^a)^{\frac{1}{a}} < (1-x^b)^{\frac{1}{a}}$ since $\frac{1}{a}$ is positive, and $y^{\frac{1}{a}}$ is an increasing function of y. Next, $(1-x^b)^{\frac{1}{a}} < (1-x^b)^{\frac{1}{b}}$ since $1-x^b < 1$, so $(1-x^b)^y$ is a decreasing function of y, and $\frac{1}{b} < \frac{1}{a}$. So $(1-x^a)^{\frac{1}{a}} < (1-x^b)^{\frac{1}{a}} < (1-x^b)^{\frac{1}{b}}$ proving the claim.

3. Show that a polynomial $p(x)$ of degree 2 which takes rational values at 3 rational values of x takes rational values at all rational x.

Let the points at which we know p is rational be x_1, x_2, and x_3. Let $q_k(x) = \frac{(x-x_i)(x-x_j)}{(x_k-x_i)(x_k-x_j)}$ where $i,j,k = 1,2,3$. Then $q_k(x)$ is a quadratic polynomial with rational coefficients that has the value 1 at x_k and 0 at the other two points. $p(x)-p(x_1)q_1(x)-p(x_2)q_2(x)-p(x_3)q_3(x)$ is a polynomial of degree at most 2 which is 0 at 3 points, so $p(x) = p(x_1)q_1(x)+p(x_2)q_2(x)+p(x_3)q_3(x)$, which has rational coefficients. This argument is really just reconstructing the Lagrange interpolation formula; the analogous argument applies to a polynomial of degree n which has rational values at $n+1$ rational points.

4. Give a necessary and sufficient condition for a 2×2 complex upper triangular matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to have an upper triangular square root.

The only such matrices without upper triangular square roots are non-zero multiples of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$$\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}^2 = \begin{pmatrix} e^2 & f(e+h) \\ 0 & h^2 \end{pmatrix}$$

One can always find at least one square root, e for a and h for d. One can then choose f to solve $f(e+h) = b$ except if b is non-zero but $e+h$ must be 0. The only way that $e+h$ must equal 0 is if both a and d are 0.

5. Let s be an arc of the unit circle lying completely in the first quadrant and of length 1. Let X be the area of the region bounded by the arc, the x-axis and the vertical lines joining the endpoints of the arc to the x-axis. Let Y be the area of the region bounded by the arc, the y-axis and the horizontal lines joining the endpoints of the arc to the y-axis. Find $X + Y$.

The answer is 1, as can be shown by a calculus calculation, but one can also do it without calculus. Draw a picture to make this clear: Let the the ends of the arc be at $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 < x_2$. Denote $A = (0, y_1)$, $B = (0, y_2)$, $C = (x_1, 0)$, $D = (x_2, 0)$, $O = (0, 0)$. Let $PQDC$ be short for the area of the region bounded by the arc PQ and the lines QD, DC, CP (so $PQDC = X$) and similarly for $PQBA = Y$ etc. Then $X = PQDC = OPQD - OPC = OPQ + OQD - OPC = OPQ + OBQ - OPC$. Similarly $Y = PQBA = OPQ + OPC - OBQ$, so $X + Y = 2OPQ = 2 \times \frac{1}{2} = 1$.

6. Prove: If f is a real valued function, continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$, which vanishes at 0 and 1, then $f'(x) = f(x)$ for some x in $(0, 1)$. (Challenge: Show that the assumption that f' is continuous is not necessary.)

Consider $g(x) = e^{-x} f(x)$. $g(0) = 0$, $g(1) = 0$, and g is differentiable on $(0, 1)$, so by Rolle's Theorem, for some $c \in (0, 1)$, $g'(c) = 0$. $0 = g'(c) = -e^{-c} f(c) + e^{-c} f'(c) = e^{-c}(f'(c) - f(c))$ so $f(c) = f'(c)$.

There are other proofs that use the continuity of f'.

7. The vertices of a regular icosahedron are

$$(0, \pm 1, \pm\alpha), (\pm\alpha, 0, \pm 1), (\pm 1, \pm\alpha, 0).$$

Find all possible values of α. (The icosahedron is the regular polyhedron with 20 triangular faces and 12 vertices.)

The possibilities are $\pm\phi, \pm\phi^{-1}$ where $\phi = \frac{1+\sqrt{5}}{2}$. Suppose $\alpha > 0$, since if α produces a regular icosahedron, so does $-\alpha$. The triangle $(0, 1, \alpha), (\alpha, 0, 1), (1, \alpha, 0)$ is always equilateral. The other point on a triangle with $(0, 1, \alpha)$ and $(1, \alpha, 0)$ may be either $(0, 1, -\alpha)$ or $(-1, \alpha, 0)$. In the first case, $4\alpha^2 = 1+(\alpha-1)^2+\alpha^2$ which reduces to $\alpha^2 = 1 - \alpha$, satisfied by $\frac{1}{\phi}$, and all edges have length 2α. In the second case, $1 + (\alpha - 1)^2 + \alpha^2 = 4$ which reduces to $\alpha^2 - \alpha = 1$, satisfied by ϕ, and all edges have length 2.

8. What is the product of the lengths of all the “diagonals” of a regular octagon inscribed in a circle of radius 1? (By a “diagonal” we mean any segment connecting two distinct vertices, so the sides of the octagon are also counted as “diagonals.”)

Let the points be the 8th roots of unity in the complex plane. Consider the product of the diagonals such that one vertex is 1. This product is the 4th root of the product of all diagonals, since there are 8 vertices and each diagonal connects 2 vertices.
Let $\zeta = e^{\frac{i\pi}{4}}$. The length of the diagonal connecting ζ^k with 1 is $|1 - \zeta^k|$.
We want to compute $(1 - \zeta)(1 - \zeta^2)...(1 - \zeta^7)$.
$(1 - \zeta^4) = (1 + 1) = 2$
$(1 - \zeta^2)(1 - \zeta^6) = (1 - i)(1 + i) = 2$
$(1 - \zeta^1)(1 - \zeta^5) = (1 - \zeta^1)(1 + \zeta^1) = (1 - \zeta^2)$
$(1 - \zeta^3)(1 - \zeta^7) = (1 - \zeta^3)(1 + \zeta^3) = (1 - \zeta^6)$
So the product we want is $|2 \times 2 \times 2| = 8$, and the complete product is $8^4 = 2^{12} = 4096$. (Alternatively, to compute $(1 - \zeta)(1 - \zeta^2)...(1 - \zeta^7)$ note that $(x-\zeta)(x-\zeta^2)...(x-\zeta^7)$ is the cyclotomic polynomial $\frac{x^8-1}{x-1} = x^7+\ldots+x+1$.)