Columbia-Barnard
MATHEMATICS PRIZE EXAM

April 5, 2001

Please print your name:

Indicate school:

[ ] First-year   [ ] Sophomore   [ ] Junior   [ ] Senior

Expected date of graduation:

This is a three-hour exam. Please print your name on each booklet that you hand in. Submit your paper even if you have done no more than one or two problems. However, work as many problems as you can because partial credit will be given for significant progress made on a problem.

Mathematics Prize Exam
Thursday, April 5, 2001

1. Anne, Barbara, and Carol are the only contestants in a race. Anne started last and during the race she swapped positions with other contestants seven times, ending the race ahead of Barbara. Who won? (Prove your answer).

2. Show that $(1-x^a)^{\frac{1}{a}} < (1-x^b)^{\frac{1}{b}}$ for all $x \in (0,1)$ if $0 < a < b$.

3. Show that a polynomial $p(x)$ of degree 2 which takes rational values at 3 rational values of x takes rational values at all rational x.

4. Give a necessary and sufficient condition for a 2×2 complex upper triangular matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to have an upper triangular square root.

5. Let s be an arc of the unit circle lying completely in the first quadrant and of length 1. Let X be the area of the region bounded by the arc, the x-axis and the vertical lines joining the endpoints of the arc to the x-axis. Let Y be the area of the region bounded by the arc, the y-axis and the horizontal lines joining the endpoints of the arc to the y-axis. Find $X+Y$.

6. Prove: If f is a real valued function, continuous on $[0,1]$ and continuously differentiable on $(0,1)$, which vanishes at 0 and 1, then $f'(x) = f(x)$ for some x in $(0,1)$. (Challenge: Show that the assumption that f' is continuous is not necessary.)

7. The vertices of a regular icosahedron are

$$(0, \pm 1, \pm \alpha), (\pm \alpha, 0, \pm 1), (\pm 1, \pm \alpha, 0).$$

Find all possible values of α. (The icosahedron is the regular polyhedron with 20 triangular faces and 12 vertices.)

8. What is the product of the lengths of all the "diagonals" of a regular octagon inscribed in a circle of radius 1? (By a "diagonal" we mean any segment connecting two distinct vertices, so the sides of the octagon are also counted as "diagonals.")

End of Exam

PROBLEMS WE DID NOT USE

9. Let $f(a) = \prod_{i=1}^{a-1} i!$. Show that $\frac{100! f(10)^2}{f(20)}$ is an integer.

10. Show that the additive group of rationals, $(\mathbb{Q}, +)$, does not have a subgroup of index 2.

11. How many subgroups does $\mathbb{Z}/60 \times \mathbb{Z}/60$ have?

12. Let $p \neq q$ be prime numbers. Show that the number of positive integers that cannot be expressed in the form $np + mq$ with $n, m \geq 0$ is $(p-1)(q-1)/2$.

13. The bottom four rows of an 8×8 chess board are filled with game pieces (32 pieces in all). A game is played in which, at each move, one is allowed to move a piece one square forward to an unoccupied square if simultaniously a horizontally adjacent piece is removed from the board (in particular, a piece can move only if it is next to a piece and the square ahead of it is unoccupied). Show that no piece can get to the eighth row of the board.

14. (Hard) Show that a polynomial that takes irrational values at all irrational values of x is constant or linear.

15. If a and b are two different real numbers and m and n are any positive integers, then for all real x,

$$\frac{1}{(x-a)^m} \times \frac{1}{(x-b)^n}$$

can be expressed, via partial fractions, as a polynomial of degree m in $1/(x-a)$ plus a polynomial of degree n in $1/(x-b)$, neither polynomial having a constant term. Find as simple a formula as you can for the coefficients of these two polynomials, expressed in terms of the degrees of the terms and the numbers a, b, m, n.

16. Let $f\colon \mathbb{N} \to \mathbb{N}$ satisfy $f(f(n)) < f(n+1)$. Show that

$$\text{(a) } f(1) = 1, \qquad \text{(b) } f(n) = n, \text{ for all } n.$$

17. Let $T(n,k)$ be the kth tower of n:

$$T(n,1) = n, \quad T(n,2) = n^n, \quad T(n,k+1) = n^{T(n,k)}.$$

For which k is $T(2,k) < T(10,10) < T(2,k+1)$?

18. The difference game is played as follows:

$$(a,b,c,d) \mapsto (|a-b|, |b-c|, |c-d|, |d-a|).$$

Show that if a,b,c,d are all integers then after some number of moves of the difference game you get $(0,0,0,0)$.