Practice Problems

Linear Algebra, Dave Bayer, February 12, 2011

These are supplemental practice problems for our first exam.

[1] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

This looks like a solution set found after row-reducing into the form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$$

we first fill in

$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & -4 & -6 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

and now compute

$$\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

giving

$$\boxed{\begin{bmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & -4 & -6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

check by plugging in for $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

[2] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

We expect $4-1=3$ equations.

Find three independent rows using the pattern $(a,b)\cdot(b,-a)=0$

$$\begin{bmatrix} 3 & -4 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = 0$$

Now plug in to find right hand side:

$$\begin{bmatrix} 3 & -4 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$$

giving

$$\boxed{\begin{bmatrix} 3 & -4 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 3 & -4 & 0 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$$

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[3] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

We solve $\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$ to find two equations.

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & -2 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

has a basis of solutions $[1\ 0\ 0\ 1]$, $[0\ 1\ 1\ 0]$.

Now plug in to find right hand side:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

giving

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

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[4] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

We combine these equations

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 1 & -1 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & -2 & 0 & -2 & -4 \\ 0 & -1 & -2 & -3 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 4 & 4 & 8 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

with solution set $\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$

check:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 6 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

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[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\underbrace{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

we plug in:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad s = t$$

so $$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 3 \\ 3 \\ -3 \end{bmatrix}$$

rescaling, $$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}$$

(Compare with 4. This is the same problem.)

[6] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

we plug in:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 10 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} \qquad \begin{aligned} s+t &= 2 \\ s &= 2-t \end{aligned}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + (2-t) \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

giving

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

(Compare with 4 5 Again, same problem.)

[7] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$(*)\quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \cancel{z}\, q \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + \cancel{z}\, r \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We set these expressions equal to each other:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + q \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} q \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

(funny column order saves steps!)

$$\left[\begin{array}{cccc|c} -2 & -1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 \\ 0 & 3 & -1 & 0 & 0 \\ -1 & -2 & 0 & -1 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} -2 & -1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 \\ -2 & 2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} -2 & -1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} -3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \end{array}\right]$$

free

giving

$$\begin{bmatrix} q \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \\ 3 \\ -3 \end{bmatrix}$$

plug back into (*)

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + u \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + u \begin{bmatrix} -3 \\ 3 \\ 3 \\ -3 \end{bmatrix}$$

rescaling,

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + u \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}$$

(Again, same problem)
see 4, 5, 6