Exam 2

Linear Algebra, Dave Bayer, Alternate, March 12, 2013

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find a basis for the set of solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ \cancel{3} & \cancel{3} & \cancel{2} & \cancel{2} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Extend this basis to a basis for $\mathbb{R}^4$.

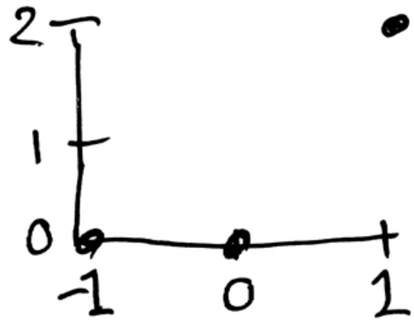
$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\left.\begin{matrix} (1,-1,0,0) \\ (0,0,1,-1) \end{matrix}\right\} \text{basis for solus}$$

$$\left.\begin{matrix} (0,1,0,0) \\ (0,0,0,1) \end{matrix}\right\} \text{extend to } \mathbb{R}^4$$

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (-1, 0), \quad (x_2, y_2) = (0, 0), \quad (x_3, y_3) = (1, 2)$$



$$\begin{array}{cc} x & 1 \end{array}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \begin{array}{l} a = 1 \\ b = 2/3 \end{array} \qquad \boxed{y = x + 2/3}$$

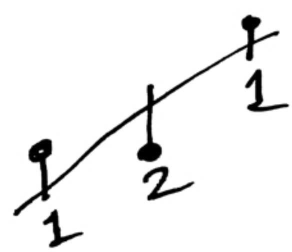
check:

x	y	est	error
-1	0	-1/3	-1
0	0	2/3	2
1	2	5/3	-1

/3

✓✓

(springs balance)



[3] Let L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects first across the line $y = x$, then across the line $y = 3x$. Find the matrix A which represents L in standard coordinates.

B reflects across $y=x$ $\quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C reflects across $y=3x$

$C\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$C\begin{bmatrix} 1 & +3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} +1 & +3 \\ +3 & -1 \end{bmatrix}\Big/10$

$= \begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix}\Big/10 = \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}\Big/5$

check: $\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}\Big/5 \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -15 \\ 15 & 5 \end{bmatrix}\Big/5 = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ ✓

$A = \overset{\text{second}}{C}\ \overset{\text{first}}{B} = \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}\Big/5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\Big/5$

$$\boxed{A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\Big/5}$$

check $(3,1) \xrightarrow[\text{(1,1)}]{\text{across}} (1,3) \xrightarrow[\text{(1,3)}]{\text{across}} (1,3)$

$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\Big/5 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}\Big/5 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ✓

[4] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,1,1), \quad (1,2,2,2), \quad (1,1,2,2), \quad (3,4,5,5)$$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ \cancel{3} & \cancel{4} & \cancel{5} & \cancel{5} \end{array} \Rightarrow \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \Rightarrow \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}$$

$$\Downarrow$$

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \quad \perp \text{ basis}$$

$$\boxed{\begin{array}{c} (1,0,0,0) \\ (0,1,0,0) \\ (0,0,1,1) \end{array}}$$

check: subspace is $y=z$

(w,x,y,z)

mutually $\perp$ ✓✓✓

belong to $y=z$ ✓✓✓

correct number? subspace dim $=3$ ✓

[5] Let V be the vector space of all polynomials of degree $\leqslant 4$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of odd polynomials: those polynomials for which $f(-x) = -f(x)$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(1-x)\,dx$$

$W = \{ax^3 + bx\}$

$v_1 = x$ $\quad \boxed{\begin{array}{l} w_1 = x \\ w_2 = 20x^3 - 6x \end{array}}$

$v_2 = x^3$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = v_2 - \left(\frac{-1/20}{-1/6}\right) w_1 = v_2 - \tfrac{6}{20} w_1$$

$$\langle v_2, w_1 \rangle = \int_0^1 x^3(1-x)\,dx = \int_0^1 (x^3 - x^4)\,dx = \tfrac{1}{4} - \tfrac{1}{5} = -1/20$$

$$\langle w_1, w_1 \rangle = \int_0^1 x(1-x)\,dx = \int_0^1 (x - x^2)\,dx = \tfrac{1}{2} - \tfrac{1}{3} = -1/6$$

rescale to $20v_2 - 6w_1 = 20x^3 - 6x = w_2$

check $\langle 20x^3 - 6x, x \rangle = \int_0^1 (20x^3 - 6x)(1-x)\,dx$

$$= \int_0^1 (20x^3 - 20x^4 - 6x + 6x^2)\,dx$$

$$= 20/4 - 20/5 - 6/2 + 6/3$$

$$= 5 - 4 - 3 + 2 = 0 \checkmark$$