Exam 2

Linear Algebra, Dave Bayer, 8:40 AM, March 12, 2013

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find a basis for the set of solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ \cancel{2} & \cancel{2} & \cancel{2} & \cancel{2} \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Extend this basis to a basis for $\mathbb{R}^4$.

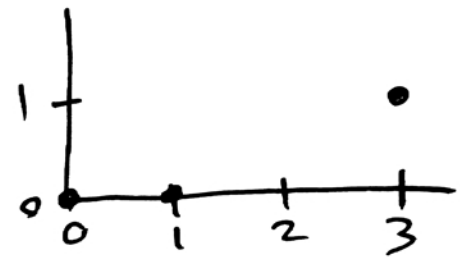
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\left.\begin{matrix} (2,0,-1,-1) \\ (0,2,-1,-1) \end{matrix}\right\} \text{basis for solus}$$

$$\left.\begin{matrix} (0,0,1,0) \\ (0,0,0,1) \end{matrix}\right\} \text{extend to } \mathbb{R}^4$$

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (0,0), \quad (x_2, y_2) = (1,0), \quad (x_3, y_3) = (3,1)$$



$$\begin{array}{cc} x & 1 \end{array} \qquad\qquad y$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix} \Big/ 14 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -2 \end{bmatrix} \Big/ 14 \qquad \boxed{y = \tfrac{5}{14}x - \tfrac{2}{14}}$$

check:

x	y	est	error
0	0	$-2/14$	-2
1	0	$3/14$	3
3	1	$13/14$	-1

$/14$

✓✓ (springs balance)

[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the line

$$x = y = 2z$$

Find the matrix A which represents L in standard coordinates.

proj (x,y,z) onto $(2,2,1)$

$$= \frac{x\,y\,z \cdot 2\,2\,1}{2\,2\,1 \cdot 2\,2\,1}(2,2,1)$$

$$= \frac{2x+2y+z}{9}(2,2,1)$$

$$= \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}\Big/9 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

x	2	2	1
2	4	4	2
2	4	4	2
1	2	2	1

$$\boxed{A = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}\Big/9}$$

check: $L(2,2,1) = (2,2,1)$
$L(1,-1,0) = (0,0,0)$
$L(0,1,-2) = (0,0,0)$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}\Big/9 \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 18 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}\Big/9$$

✓

[4] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ given by the equation $w + x + y - 2z = 0$.

$$\left.\begin{matrix}(1,-1,0,0)\\(0,0,2,1)\end{matrix}\right\}$$ two $\perp$ vectors in subspace

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$(r, r, s, -2s)$ is $\perp$ to both
$(1,-1,0,0)$
$(0,0,2,1)$

$(1,1,1,-2) \cdot (r,r,s,-2s)$
$= 2r + 5s = 0$
$r = 5, s = -2$
$(5,5,-2,4)$

$$\boxed{\begin{matrix}(1,-1,0,0)\\(0,0,2,1)\\(5,5,-2,4)\end{matrix}}$$

check: mutually orthogonal? ✓✓✓

all satisfy $\underbrace{w+x+y}_{\text{first three coords add to twice last}} - 2z = 0$ ✓✓✓

correct number?
Subspace dim = 3 ✓

[5] Let V be the vector space of all polynomials of degree $\leqslant 3$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(0) = f(1) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$V = \{ ax^3 + bx^2 + cx + d \}$$

$$f(0) = d = 0$$
$$f(1) = a + b + c + d = 0$$

$$\underbrace{\begin{aligned} v_1 &= x^2 - x \\ v_2 &= x^3 - x \end{aligned}}_{f(0)=f(1)=0 \ ✓ \ ✓} \Rightarrow \boxed{\begin{aligned} w_1 &= x^2 - x \\ w_2 &= x^3 - 3/2\,x^2 + 1/2\,x \end{aligned}}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 - 3/2\, w_1 = \begin{array}{l} x^3 - x \\ -3/2\,x^2 + 3/2\,x \end{array}$$

$$\langle v_2, w_1\rangle = \int_0^1 (x^3 - x)(x^2 - x)\,dx = \int_0^1 (x^5 - x^4 - x^3 + x^2)\,dx$$
$$= \frac{1}{6} - \frac{1}{5} - \frac{1}{4} + \frac{1}{3}$$
$$= (10 - 12 + 15 + 20)/60$$
$$= 3/60$$

$$\langle w_1, w_1\rangle = \int_0^1 (x^2 - x)(x^2 - x)\,dx = \int_0^1 (x^4 - x^3 - x^3 + x^2)\,dx$$
$$= \frac{1}{5} - \frac{1}{4} - \frac{1}{4} + \frac{1}{3}$$
$$= (12 - 15 - 15 + 20)/60$$
$$= 2/60$$

Check: $\langle x^2 - x, 2x^3 - 3x^2 + x\rangle$

$$= \int_0^1 (x^2 - x)(2x^3 - 3x^2 + x)\,dx$$
$$= \int_0^1 (2x^5 - 3x^4 + x^3 - 2x^4 + 3x^3 - x^2)\,dx = 2/6 - 3/5 + 1/4 - 2/5 + 3/4 - 1/3$$
$$= (\cancel{20} - \cancel{36} + \cancel{15} - \cancel{24} + \cancel{45} - \cancel{20})/60 \ ✓$$