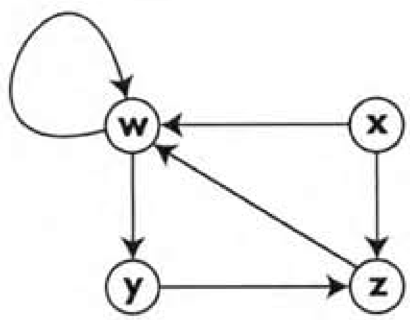
Exam 1 Solutions

Linear Algebra, Section 002 (TR 8:40am – 9:55am), Dave Bayer, February 12, 2013

[1]	[2]	[3]	[4]	[5]	Total

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Using matrix multiplication, count the number of paths of length eight from w to itself.



(Your graph may vary, to discourage and detect copying.)

$$\begin{array}{cc} & \overbrace{w\quad x\quad y\quad z}^{\text{in}} \\ \text{out}\left\{\begin{array}{c}w\\x\\y\\z\end{array}\right. & \begin{bmatrix}1&1&0&1\\0&0&0&0\\1&0&0&0\\0&1&1&0\end{bmatrix}\end{array} = A$$

$$A^2 = AA = \begin{bmatrix}1&2&1&1\\0&0&0&0\\1&1&0&1\\1&0&0&0\end{bmatrix} \qquad A^4 = A^2A^2 = \begin{bmatrix}3&3&1&2\\0&0&0&0\\2&2&1&1\\1&2&1&1\end{bmatrix}$$

$$A^8 = A^4A^4 = \begin{bmatrix}13 & \cdots \\ \vdots & \\ & \end{bmatrix}$$

don't need other entries

$\boxed{\text{13 paths}}$

[1] Check:

no path can use (x)

How many times do we loop (w)(y)(z)(w) ?

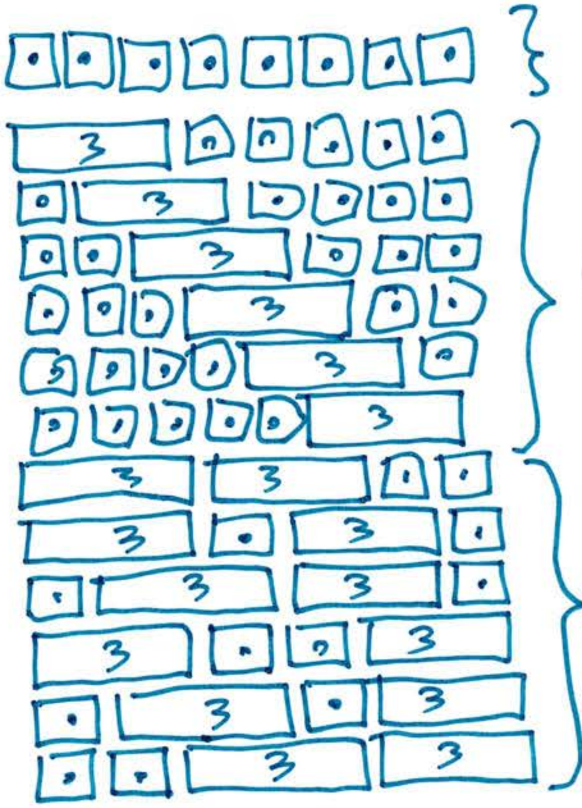
$\boxed{\text{3 steps}}$

Room for 0, 1, or 2 such loops

0 loops: 1

1 loop: 6

2 loops: 6



$\boxed{1+6+6 = 13}$

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1 loop is 5 balls in two bins

|•••••, •|••••, ••|•••, •••|••, ••••|•, •••••|

6 positions, 1 divider, where do you put the divider?

$\binom{6}{1} = 6$ ways

Similarly, 2 loops is 2 balls in 3 bins

||••, |•|•, •||•, |••|, •|•|, ••||

4 positions, 2 dividers, where do you put the divider?

$\binom{4}{2} = \frac{4\cdot 3}{2\cdot 1} = \frac{12}{2} = 6$ ways

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[2] Solve the following system of equations.

$$\begin{bmatrix} 2 & 0 & 1 & 4 \\ 3 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

controlled

$$\begin{bmatrix} 2 & 0 & 1 & 4 \\ 3 & 1 & 0 & 5 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 7 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & -5 \\ -2 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

free

particular

homogeneous

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & -5 \\ -2 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}$$

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[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} -6 & 1 \\ 3 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -6 & 1 \\ 3 & 0 \end{bmatrix} \underset{①\leftarrow ①+2②}{\Longrightarrow} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \underset{②\leftarrow \frac{1}{3}②}{\Longrightarrow} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underset{①\leftrightarrows②}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\underset{①\leftarrow ①-2②}{\Longleftarrow} \qquad \underset{②\leftarrow 3②}{\Longleftarrow} \qquad \underset{①\leftrightarrows②}{\Longleftarrow}$$

$$A = \left(\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) I$$

$$\boxed{A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

(Be explicit what you consider the answer to be. Making us guess, or choose between alternatives, will cost points.)

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} -6 & 1 \\ 3 & 0 \end{bmatrix}$$

check ✓

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[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 3 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & -3 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular}} + \underbrace{\begin{bmatrix} -2 & -2 \\ 3 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{homogeneous}} \begin{bmatrix} s \\ t \end{bmatrix} \Bigg) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

[1] use homogeneous to fill in

[2] use particular to fill in

$$\boxed{\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

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[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix} \Big/ 11 \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -11 \\ 11 \end{bmatrix} \Big/ 11 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

check: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ ✓