Exam 2, 9:10-10:25

Linear Algebra, Dave Bayer, March 27, 2012

[1] Let V be the vector space of all polynomials $f(x)$ of degree $\leqslant 2$. Find a basis for the subspace W defined by

$$f(1) = 0.$$

Extend this to a basis for V.

Usual basis for $V = \{x^2, x, 1\}$

$V = \{ax^2+bx+c\} \leftrightarrow (a,b,c) \in \mathbb{R}^3$

$f(x) = ax^2+bx+c$

$f(1) = a+b+c = 0$ defines a plane in $\mathbb{R}^3$

dim=2, expect 2 basis vectors

$$\begin{matrix}\text{independent,}\\ \text{start in}\\ \text{different}\\ \text{columns}\end{matrix}\left\{\begin{array}{l}\left.\begin{array}{l}(1,-1,0) \leftrightarrow x^2-x\\ (0,1,-1) \leftrightarrow x-1\end{array}\right\} W\\ \hline (0,0,1) \leftrightarrow 1 \quad \text{extends to } V\end{array}\right.$$

$$\boxed{\begin{matrix}\text{basis for}\\ V\end{matrix}\left\{\begin{array}{c}\left.\begin{array}{c}x^2-x\\ x-1\end{array}\right\}\text{basis for } W\\ \hline 1\end{array}\right.}$$

(Use polynomial notation, for answer. vector notation only for intermediate calculations)

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

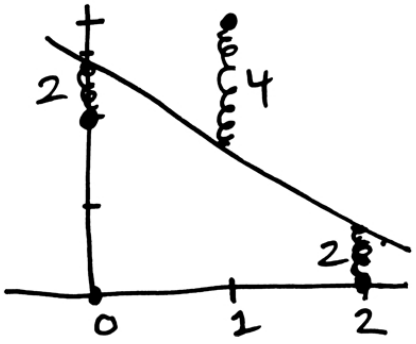
$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \end{bmatrix} \Big/ 6$$

so $a = -1$, $b = 8/3$

$$\boxed{y = -x + 8/3}$$

Check:



x	y	$-x+8/3$	Δ
0	2	$8/3$	$-2/3$
1	3	$5/3$	$4/3$
2	0	$2/3$	$-2/3$

[3] Let V be the subspace of $\mathbb{R}^5$ spanned by the vectors

$$(1,1,1,0,0), \quad (0,0,1,1,1).$$

Find an orthogonal basis for the subspace V.

$v_1=(1,1,1,0,0)$ $\qquad w_1=(1,1,1,0,0)$

$v_2=(0,0,1,1,1)$ $\qquad w_2 = v_2 - \frac{v_2\cdot w_1}{w_1\cdot w_1}\, w_1$

$$= (0,0,1,1,1) - \frac{00111\cdot 11100}{11100\cdot 11100}(1,1,1,0,0)$$

$$= (0,0,1,1,1) - \tfrac{1}{3}(1,1,1,0,0)$$

rescale to $\quad 3(0,0,1,1,1)-(1,1,1,0,0)$

$$= (-1,-1,2,3,3)$$

$$\boxed{\{(1,1,1,0,0),\ (-1,-1,2,3,3)\}}$$

Check $w_1\cdot w_2=0$ ✓

Need 3 equs on span $\quad (a,b,c,d,e)$

$a=b \quad d=e$

$c=b+d$

$(1,1,1,0,0)$ ✓✓✓

$(0,0,1,1,1)$ ✓✓✓

$$\text{Now}\quad (\underbrace{-1,-1}_{\substack{a=b\\ ✓}}\overbrace{{}\,,2,\,{}}^{c=b+d\ ✓}\underbrace{3,3}_{\substack{d=e\\ ✓}})$$

[4] Using Cramer's rule, solve for z in

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$z = \frac{\begin{vmatrix} 1 & 0 & 0 & \boxed{0} \\ 1 & 3 & 0 & \boxed{0} \\ 1 & 3 & 3 & \boxed{0} \\ 1 & 3 & 3 & \boxed{1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 3 & 3 & 3 \end{vmatrix}} = \frac{9}{18} = \frac{1}{2}$$

elementary column op ④←④−①

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 3 & 3 & 2 \end{vmatrix} = 1 \cdot 3 \cdot 3 \cdot 2 = 18$$

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check

$$x: \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 3 & 3 & 1 \\ \textcircled{1} & 3 & 3 & 3 \end{vmatrix} \qquad y: \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & \textcircled{1} & 3 & 3 \end{vmatrix} \qquad z: \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 3 & \textcircled{1} & 3 \end{vmatrix}$$

$$-\frac{1}{2} \qquad\qquad 0 \qquad\qquad 0$$

$$-\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad ✓$$

[5] Let V be the subspace of $\mathbb{R}^4$ consisting of all solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Find the matrix A which projects orthogonally onto the subspace V.

$W \perp V$ $\quad W$ basis given by rows which are $\perp$ $\quad (1,1,-1,-1)$, $(1,-1,-1,1)$

So projection to V is

$$(1,0,0,0) \mapsto (1,0,0,0) - \frac{1000\cdot 11{-}1{-}1}{11{-}1{-}1\cdot 11{-}1{-}1}(1,1,-1,-1) - \frac{1000\cdot 1{-}1{-}11}{1{-}1{-}11\cdot 1{-}1{-}11}(1,-1,-1,1)$$

$$= (1,0,0,0) - \tfrac{1}{4}(1,1,-1,-1) - \tfrac{1}{4}(1,-1,-1,1)$$

$$= (\tfrac{1}{2},0,\tfrac{1}{2},0)$$

$$(0,1,0,0) \mapsto (0,1,0,0) - \tfrac{1}{4}(1,1,-1,-1) + \tfrac{1}{4}(1,-1,-1,1)$$

$$= (0,\tfrac{1}{2},0,\tfrac{1}{2})$$

$$(0,0,1,0) \mapsto (0,0,1,0) + \tfrac{1}{4}(1,1,-1,-1) + \tfrac{1}{4}(1,-1,-1,1)$$

$$= (\tfrac{1}{2},0,\tfrac{1}{2},0)$$

$$(0,0,0,1) \mapsto (0,0,0,1) + \tfrac{1}{4}(1,1,-1,-1) - \tfrac{1}{4}(1,-1,-1,1)$$

$$= (0,\tfrac{1}{2},0,\tfrac{1}{2})$$

so $$\boxed{A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} /2}$$