F17 10:10 Exam 1 Problem 1
Linear Algebra, Dave Bayer

test1b1p1

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Test 1

Name Solutions Uni

[1] Find the general solution to the following system of equations.

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

1, 2, 3 independent

5 vars − 3 conditions = 2-dimensional solution

$$\boxed{\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}$$

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

Rows 1 and 2 are not multiples of each other, so they span a plane. Anything in this plane has first two coordinates equal to each other, which is not true for row 3, so row 3 is independent of 1, 2.

3 conditions on 5 variables leaves a 5−3=2 dimensional solution space, so answer must be of the form

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ast \\ \ast \\ \ast \\ \ast \\ \ast \end{bmatrix} + \underbrace{\begin{bmatrix} \ast & \ast \\ \ast & \ast \\ \ast & \ast \\ \ast & \ast \\ \ast & \ast \end{bmatrix}}_{\text{2 columns for 2 parameters}} \begin{bmatrix} s \\ t \end{bmatrix} \Leftarrow \text{2 parameters for 2 dimensions}$$

This analysis is the most important part of the problem.
If your answer doesn't have this shape, it gives the impression that you don't understand the problem at all.

$[b|p|]\ldots$

Particular solution: Find (v,w,x,y,z) so

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

expand:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} v + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} w + \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} y}_{} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} z = \cancel{0}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$(0,0,-1,1,0)$ | $-\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

so $x=-1, y=1$

Homogeneous solutions:

$$\begin{array}{c|ccccccc} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & & \\ (1,0,0,0,-1) & (1) & & & & (-1) & = & 0 \\ (0,1,1,1,-4) & & (1) & (1) & (1) & (-4) & = & 0 \end{array}$$

$$\underbrace{(1)\ (1)\ (1)}_{\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}} \qquad \underbrace{(-4)}_{\begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}}$$

It is essential that our basic solutions are independent. Take two multiples of the same vector (such as its negative) is a wrong answer.

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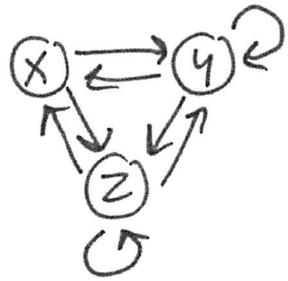
Test 1

[2] Using matrix multiplication, count the number of paths of length five from x to z.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{ccc} x & y & z \end{array} \\ \text{to} \; \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ & A \end{array}$$



number of paths = 44

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number of paths = 44

$$A^2 = \underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{A^2}{\begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}}$$

$$A^4 = \underset{A^2}{\begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}} \underset{A^2}{\begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}} = \underset{A^4}{\begin{bmatrix} \boxed{\begin{matrix} 12 \\ 16 \\ 16 \end{matrix}} & \end{bmatrix}}$$

$$A^5 = \underset{A}{\begin{bmatrix} & & \\ \hline 1 & 1 & 1 \end{bmatrix}} \underset{A^4}{\begin{bmatrix} \boxed{\begin{matrix} 12 \\ 16 \\ 16 \end{matrix}} & \end{bmatrix}} = \underset{A^5}{\begin{array}{cc} & x \\ & \begin{bmatrix} & \\ \boxed{44} & \end{bmatrix} \\ z & \end{array}}$$

need these entries

need this entry

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test1b1p3

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Test 1

[3] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix}}_{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}}, \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} t$$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$-s + 2t = 2$$

solve like any system

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

then plug in:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u\right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

check also in left space:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ 1+u \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u \quad \checkmark$$

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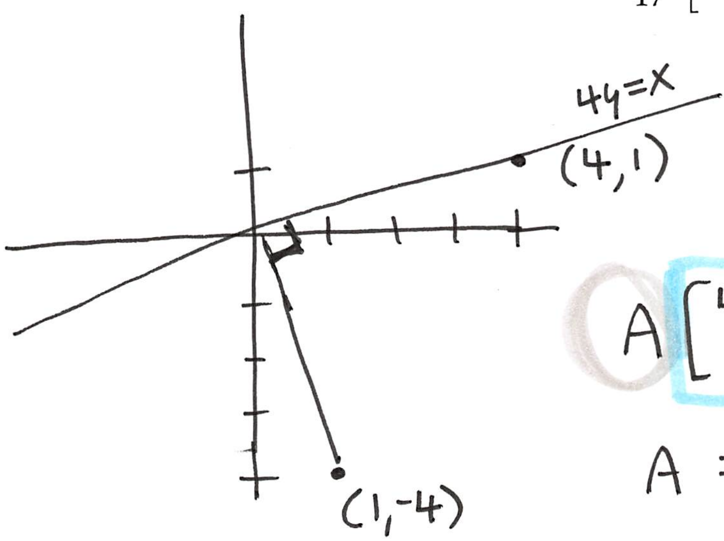
Test 1

[4] Find the 2×2 matrix A that reflects across the line $4y = x$.

$$A = \frac{1}{\boxed{17}} \begin{bmatrix} \boxed{15} & \boxed{8} \\ \boxed{8} & \boxed{-15} \end{bmatrix}$$

$$A = \frac{1}{17} \begin{bmatrix} 15 & 8 \\ 8 & -15 \end{bmatrix}$$



$(4,1) \longmapsto (4,1)$

$(1,-4) \longmapsto (-1,4)$

$$A \boxed{\begin{bmatrix} 4 & 1 \\ 1 & -4 \end{bmatrix}} = \boxed{\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}}$$

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -4 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} +4 & +1 \\ +1 & -4 \end{bmatrix} / 17$$

$$= \begin{bmatrix} 15 & 8 \\ 8 & -15 \end{bmatrix} / 17$$

check: $\underbrace{\begin{bmatrix} 15 & 8 \\ 8 & -15 \end{bmatrix} / 17}_{A} \boxed{\begin{bmatrix} 4 & 1 \\ 1 & -4 \end{bmatrix}} = \begin{bmatrix} 68 & -17 \\ 17 & 68 \end{bmatrix} / 17 = \boxed{\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}}$ ✓

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Test 1

[5] Find a basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$$\underset{u}{(1,1,-1,-1)},\quad \underset{v}{(1,-1,1,-1)},\quad \underset{w}{(1,-1,-1,1)},\quad \underbrace{\underset{-w}{(-1,1,1,-1)},\quad \underset{-v}{(-1,1,-1,1)},\quad \underset{-u}{(-1,-1,1,1)}}_{\text{redundant}}$$

$$\text{average}(u,v) = (1,0,0,-1)$$
$$\text{average}(u,w) = (1,0,-1,0)$$
$$\text{average}(v,w) = (1,-1,0,0)$$

these end in different positions, hence independent, So subspace is dimension 3

$$\boxed{\text{Basis is}\quad \begin{aligned} u &= (1,1,-1,-1) \\ v &= (1,-1,1,-1) \\ w &= (1,-1,-1,1) \end{aligned}}$$

Note this problem does <u>not</u> ask to extend basis to $\mathbb{R}^4$!