F15 Final Exam Problem 1
Linear Algebra, Dave Bayer

test1a4p1

[Reserved for Score]

Test 1

Name ______________________ Uni ____________

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \qquad\qquad\qquad\qquad}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

F15 Final Exam Problem 2
Linear Algebra, Dave Bayer

test1a4p2

[Reserved for Score]

Test 1

[2] Find the 3×3 matrix A such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$A = \frac{1}{\boxed{}} \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

$$A = \frac{1}{7} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -3 \\ -4 & -2 & 6 \end{bmatrix}$$

F15 Final Exam Problem 3
Linear Algebra, Dave Bayer

test1a4p3

[Reserved for Score]

Test 1

[3] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Find $f(8)$.

$f(8) = \boxed{}$

$$f(8) = 34$$

F15 Final Exam Problem 4
Linear Algebra, Dave Bayer

test1a4p4

[Reserved for Score]

Test 1

[4] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 0 \end{bmatrix}$$

$$\boxed{e^{At} = \frac{\square}{\square}\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} + \frac{\square}{\square}\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}}$$

$$\lambda = -2, 3 \qquad e^{At} = \frac{e^{-2t}}{5}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} + \frac{e^{3t}}{5}\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$$

F15 Final Exam Problem 5
Linear Algebra, Dave Bayer

test1a4p5

[Reserved for Score]

Test 1

[5] Find A^n where A is the matrix

$$A \;=\; \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^n \;=\; \frac{\square}{\square}\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} + \frac{\square}{\square}\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} + \frac{\square}{\square}\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

$$\lambda = 1, 2, 3 \qquad A^n \;=\; \frac{1^n}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} + 2^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix} + \frac{3^n}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 3 & 0 \end{bmatrix}$$

F15 Final Exam Problem 6
Linear Algebra, Dave Bayer

test1a4p6

[Reserved for Score]

Test 1

[6] Solve the differential equation $y' = Ay$ where

$$A \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \qquad y(0) \;=\; \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{y \;=\; \frac{\square}{\square}\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} + \frac{\square}{\square}\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix} + \frac{\square}{\square}\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}}$$

$$\lambda = 3, 0, 0 \qquad e^{At} \;=\; \frac{e^{3t}}{9}\begin{bmatrix} 4 & 2 & 3 \\ 4 & 2 & 3 \\ 4 & 2 & 3 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 5 & -2 & -3 \\ -4 & 7 & -3 \\ -4 & -2 & 6 \end{bmatrix} + \frac{t}{3}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$y \;=\; \frac{e^{3t}}{9}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -5 \\ 4 \\ 4 \end{bmatrix} + \frac{t}{3}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

F15 Final Exam Problem 7
Linear Algebra, Dave Bayer

test1a4p7

[Reserved for Score]

Test 1

[7] Express the quadratic form

$$2x^2 \;+\; 2y^2 \;-\; 2xz \;+\; 2yz \;+\; 3z^2$$

as a sum of squares of othogonal linear forms.

$$\square\left(\qquad\right)^2 + \square\left(\qquad\right)^2 + \square\left(\qquad\right)^2$$

$$\lambda = 1, 2, 4 \qquad A \;=\; \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$$

$$\frac{1}{3}\,(x \;-\; y \;+\; z)^2 \;+\; (x \;+\; y)^2 \;+\; \frac{2}{3}\,(x \;-\; y \;-\; 2z)^2$$

F15 Final Exam Problem 8
Linear Algebra, Dave Bayer

Test 93

test93a4p8

[Reserved for Score]

[8] Solve for z in the system of differential equations

$$\begin{aligned} y'' &= 2y' + y + z \\ z' &= -2y' + 2y + z \end{aligned}$$

where

$$y(0) = y'(0) = 0, \qquad z(0) = 1$$

$$\begin{bmatrix} y'' \\ y' \\ z' \end{bmatrix} = \overbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}}^{A} \begin{bmatrix} y' \\ y \\ z \end{bmatrix}$$

$$\boxed{z(t) = e^t - t^2 e^t}$$

$r+s+t = 2+0+1 = 3$

$rs+rt+st = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} = -1+4+0 = 3$

$rst = -1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$

$\Rightarrow r,s,t = 1,1,1$

$$A - 1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix} \qquad (A-1)^3 = 0$$

$$\begin{bmatrix} y' \\ y \\ z \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \underset{(**)}{=} e^t e^{(A-1)t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^t\left(I + (A-1)t + (A-1)^2 t^2/2\right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y' \\ y \\ z \end{bmatrix} = e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \tfrac{1}{2} t^2 e^t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\boxed{z = e^t - t^2 e^t}$$

Check

	e^t	te^t	$\frac{1}{2}t^2e^t$		
y	0	0	1	1	2
y'	0	1	1	2	-2
y''	1	2	1		
z	1	0	-2	1	1
z'	1	-2	-2		
y''	1	2	1		
z'	1	-2	-2		

(**) $A = I + (A - I)$

$e^{At} = e^{It} e^{(A-I)t}$

if $N^3 = 0$ then

$$e^N = I + N + N^2/2 + \underbrace{N^3/6 + N^4/24 + \cdots}_{0}$$