$\boxed{3A1}$

$$A=\begin{bmatrix}2&0&0\\2&1&1\\1&2&2\end{bmatrix} \qquad A-\lambda I=\begin{bmatrix}\boxed{2-\lambda}&0&0\\2&1-\lambda&1\\1&2&2-\lambda\end{bmatrix}$$

block triangular

eigenvalues are 2, and those of $\begin{bmatrix}1&1\\2&2\end{bmatrix}$

$$\left.\begin{array}{l}\text{sum}=\text{trace}=3\\ \text{prod}=\det=0\end{array}\right\}\Rightarrow 0,3 \qquad \lambda=0,2,3$$

$$\lambda=0:\begin{bmatrix}2&0&0\\2&1&1\\1&2&2\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix}=0 \quad \lambda=2:\begin{bmatrix}0&0&0\\2&-1&1\\1&2&0\end{bmatrix}\begin{bmatrix}2\\-1\\-5\end{bmatrix}=0 \quad \lambda=3:\begin{bmatrix}-1&0&0\\2&-2&1\\1&2&-1\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}=0$$

$$A=\underset{S\leftarrow V}{\begin{bmatrix}0&2&0\\1&-1&1\\-1&-5&2\end{bmatrix}}\underset{V\leftarrow V}{\begin{bmatrix}0&&\\&2&\\&&3\end{bmatrix}}\underset{V\leftarrow S}{\begin{bmatrix}3&-4&2\\-3&0&0\\-6&-2&-2\end{bmatrix}}/(-6)$$

$$\begin{array}{c|cccc} & 0 & 1 & -1 & 0\ \ 1 \\ \hline 2 & -1 & -5 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 2 & -1 & -5 & 2 & -1 \end{array}$$

$$\begin{array}{c|ccc} & -3 & 4 & -2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & -3 & 4 & -2 \\ -1 & 3 & -4 & 2 \end{array}/6 \qquad \begin{array}{c|ccc} & 3 & 0 & 0 \\ \hline 2 & 6 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ -5 & -15 & 0 & 0 \end{array}/6 \qquad \begin{array}{c|ccc} & 6 & 2 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 6 & 2 & 2 \\ 2 & 12 & 4 & 4 \end{array}/6$$

$$\Rightarrow \boxed{A^n=\frac{0^n}{6}\begin{bmatrix}0&0&0\\-3&4&-2\\3&-4&2\end{bmatrix}+\frac{2^n}{2}\begin{bmatrix}2&0&0\\-1&0&0\\-5&0&0\end{bmatrix}+\frac{3^n}{3}\begin{bmatrix}0&0&0\\3&1&1\\6&2&2\end{bmatrix}}$$

check $n=2$:

$$4\begin{bmatrix}6&0&0\\-3&0&0\\-15&0&0\end{bmatrix}/6+9\begin{bmatrix}0&0&0\\6&2&2\\12&4&4\end{bmatrix}/6=\begin{bmatrix}24&0&0\\42&18&18\\48&36&36\end{bmatrix}/6=\begin{bmatrix}4&0&0\\7&3&3\\8&6&6\end{bmatrix} ✓$$

$\boxed{3B1}$

$$A=\begin{bmatrix} \boxed{\begin{matrix}2&1\\2&1\end{matrix}} & \begin{matrix}1\\2\end{matrix} \\ \begin{matrix}0&0\end{matrix} & \boxed{2}\end{bmatrix} \qquad \lambda=2,\ \text{roots of } \begin{bmatrix}2&1\\2&1\end{bmatrix} \quad \begin{matrix}\text{trace } 3\\ \det 0\end{matrix}$$

$$\lambda = 0, 2, 3$$

$\lambda=0$ $\qquad\qquad$ $\lambda=2$ $\qquad\qquad$ $\lambda=3$

$$\begin{bmatrix}2&-2&1\end{bmatrix}\begin{bmatrix}2&1&1\\2&1&2\\0&0&2\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix} \qquad \underbrace{\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\2&-1&2\\0&0&0\end{bmatrix}}_{0}\begin{bmatrix}-3\\-2\\+2\end{bmatrix} \qquad \begin{bmatrix}2&1&4\end{bmatrix}\begin{bmatrix}-1&1&1\\2&-2&2\\0&0&-1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\overbrace{\begin{bmatrix}0&1&1\\2&-1&2\\0&0&0\end{bmatrix}\begin{bmatrix}-3\\-2\\+2\end{bmatrix}}^{0}$$

$$\begin{array}{c|ccc} & 2 & -2 & 1 \\ \hline 1 & 2 & -2 & 1 \\ -2 & -4 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{array} \Big/ 6 \qquad \begin{array}{c|ccc} & 0 & 0 & 1 \\ \hline -3 & 0 & 0 & -3 \\ -2 & 0 & 0 & -2 \\ 2 & 0 & 0 & 2 \end{array} \Big/ 2 \qquad \begin{array}{c|ccc} & 2 & 1 & 4 \\ \hline 1 & 2 & 1 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \Big/ 3$$

$$6 = (1,-2,0)\cdot(2,-2,1) \qquad 2 = (-3,-2,2)\cdot(0,0,1) \qquad 3 = (1,1,0)\cdot(2,1,4)$$

$$e^{At} = \frac{1}{6}\begin{bmatrix}2&-2&1\\-4&4&-2\\0&0&0\end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix}0&0&-3\\0&0&-2\\0&0&2\end{bmatrix} + \frac{e^{3t}}{3}\begin{bmatrix}2&1&4\\2&1&4\\0&0&0\end{bmatrix}$$

check $$e^{At}\Big|_{t=0} = \begin{bmatrix}2&-2&1\\-4&4&-2\\0&0&0\end{bmatrix}\Big/6 + \begin{bmatrix}0&0&-9\\0&0&-6\\0&0&+6\end{bmatrix}\Big/6 + \begin{bmatrix}4&2&8\\4&2&8\\0&0&0\end{bmatrix}\Big/6 = I \ ✓$$

$$\left(\frac{d}{dt}e^{At}\right)\Big|_{t=0} = 0\begin{bmatrix}\ &\ &\ \end{bmatrix} + 2\begin{bmatrix}0&0&-9\\0&0&-6\\0&0&6\end{bmatrix}\Big/6 + 3\begin{bmatrix}4&2&8\\4&2&8\\0&0&0\end{bmatrix}\Big/6 = \begin{bmatrix}12&6&6\\12&6&12\\0&0&12\end{bmatrix}\Big/6 = A \ ✓$$

$\boxed{3C1}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$r+s+t = \text{trace}(A) = 3$

$rs+rt+st = \text{huh}(A)$

$= \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1+1-1 = -1$

$rst = \det(A) = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -3$

-3	-1	1	3
	x	x	x

$(-1)+1+3=3$ ✓

$(-1)1+(-1)3+1\cdot 3 = -1$ ✓

$(-1)1\cdot 3 = -3$ ✓

$\lambda = -1, 1, 3$

$\lambda=-1$: $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 0$

$\lambda=1$: $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = 0$

$\lambda=3$: $\begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & 2 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 0$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{-1}{2}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$ } express initial condition in eigenbasis

$$\boxed{y = \frac{e^t}{2}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{e^{3t}}{2}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}$$

check $y' = \frac{e^t}{2}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{3e^{3t}}{2}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$= Ay = \frac{e^t}{2}\underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} ✓} + \frac{e^{3t}}{2}\underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} ✓}$$

$\boxed{3D1}$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$r+s+t = \mathrm{trace}(A) = 5$

$rs+rt+st = \mathrm{huh}(A)$

$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 1+4+2 = 7$

$rst = \det(A) = 2\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3$

cols sum to 3

$\Rightarrow$ $[1\ 1\ 1]$ is left eigenvector with $\lambda = 3$

$3+s+t = 5 \Rightarrow s+t = 2$ $\quad \lambda = 1, 1, 3$ repeated root

$3st = 3 \Rightarrow st = 1$

$$B_1 = A - 1I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad B_3 = A - 3I = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$B_1^2 = (A-1I)^2 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

(for repeated root)

$$\underset{v_2}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \xmapsto{B_1} \underset{v_1}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \xmapsto{B_1} 0 \qquad \underset{v_3}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} \xmapsto{B_3} 0$$

$$\Rightarrow \begin{aligned} Av_2 &= v_1 + v_2 \\ Av_1 &= v_1 \\ Av_3 &= 3v_3 \end{aligned} \quad \Rightarrow \underset{V \leftarrow V}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}^n} = \begin{bmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{bmatrix}$$

$$A = \underset{S \leftarrow V}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}} \underset{V \leftarrow V}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}} \underset{V \leftarrow S}{\begin{bmatrix} -1 & -1 & 3 \\ -2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}} \Big/ (-4)$$

$$\begin{array}{c|cccc} 0 & 1 & -1 & 0 & 1 \\ \hline 1 & -1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 \end{array}$$

$$\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = I - \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

$$\begin{array}{c|ccc} & 2 & -2 & -2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 2 & -2 & -2 \\ -1 & -2 & 2 & 2 \end{array} \Big/ 4 \qquad \begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \Big/ 4 \qquad \begin{bmatrix} 2 & -2 & -2 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \Big/ 4$$

m3D1

$$\boxed{A^n = \frac{1}{4}\begin{bmatrix} 2 & -2 & -2 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} + \frac{n}{4}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & -2 \\ -2 & 2 & 2 \end{bmatrix} + \frac{3^n}{4}\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

check: $n=0$: $$\begin{bmatrix} 2 & -2 & -2 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}/4 + \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}/4 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}/4 = I \checkmark$$

$n=1$: $$\begin{bmatrix} 2 & -2 & -2 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}/4 + \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & -2 \\ -2 & 2 & 2 \end{bmatrix}/4 + \begin{bmatrix} 6 & 6 & 6 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}/4 = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 4 & 0 \\ 0 & 4 & 8 \end{bmatrix}/4 = A \checkmark$$

$\boxed{\text{3E1}}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \boxed{\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}} \\ 2 & & \end{bmatrix}$ $\lambda = 1$, roots of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ sum 4, prod 3 $\Rightarrow 1, 3$

$\lambda = 1, 1, 3$

---

suppose instead $A = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 3 \end{bmatrix}$ (as in nice coords)

$A - 1 = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 2 \end{bmatrix}$ $(A-1)^2 = \begin{bmatrix} 0 & 0 & \\ & 0 & \\ & & 4 \end{bmatrix}$ $\frac{(A-1)^2}{4} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} = B$

$I - \frac{(A-1)^2}{4} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = C = I - B$

$A - C - 3B = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} = D$

$$e^{At} = e^t C + te^t D + e^{3t} B$$

these formulas depend only on eigenvalues, hold in any coord system.

Use our coords:

$$B = \frac{(A-I)^2}{4} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}^2 \Big/ 4 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix} \Big/ 4$$

$$C = I - B = \begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix} \Big/ 4 - \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix} \Big/ 4 = \begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & -2 \\ -3 & -2 & 2 \end{bmatrix} \Big/ 4$$

$$D = A - C - 3B = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 8 & 4 \\ 8 & 4 & 8 \end{bmatrix} \Big/ 4 - \begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & -2 \\ -3 & -2 & 2 \end{bmatrix} \Big/ 4 - \begin{bmatrix} 0 & 0 & 0 \\ 9 & 6 & 6 \\ 9 & 6 & 6 \end{bmatrix} \Big/ 4 = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \Big/ 4$$

$$e^{At} = e^t \begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & -2 \\ -3 & -2 & 2 \end{bmatrix} \Big/ 4 + te^t \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Big/ 2 + e^{3t} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix} \Big/ 4$$

$\boxed{1,1,3E1}$

check: $A^n = \begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & -2 \\ -3 & -2 & 2 \end{bmatrix}/4 + n\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}/4 + 3^n\begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix}/4$

$n=0$: $\begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & -2 \\ -3 & -2 & 2 \end{bmatrix}/4 + \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix}/4 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}/4 = I$ ✓

$n=1$ $\begin{bmatrix} 4 & 0 & 0 \\ -3 & 2 & -2 \\ -3 & -2 & 2 \end{bmatrix}/4 + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}/4 + \begin{bmatrix} 0 & 0 & 0 \\ 9 & 6 & 6 \\ 9 & 6 & 6 \end{bmatrix}/4 = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 8 & 4 \\ 8 & 4 & 8 \end{bmatrix}/4 = A$ ✓

$\boxed{3F1}$ $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ & & 1 \end{bmatrix}$ roots $1, \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$\lambda = 1, 1, 3$

$A-3I$: $\underbrace{[1\ 1\ 1]\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ & & -2 \end{bmatrix}}_{0}$ $\overbrace{\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ & & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}^{0}$

$$\begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} /2$$

so $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}/2$ projects to subspace where $A=3$, $A-3=0$

check: $B^2 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}/4 = B$ ✓

$AB = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}/2 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}/2 = 3B$ ✓

Therefore $I-B = C$ projects to space where $(A-I)^2 = 0$

$C = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}/2 - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}/2 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}/2$

check: $C^2 = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}/4 = C$ ✓

$A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ already $(A-I)C = 0$. huh.
$\Rightarrow$ don't need $(A-I)^2$!!

so

$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}/2 + 3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}/2 = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}/2$ ✓

$\boxed{3F1_{iii}}$ so

$$y = e^{At}\begin{bmatrix}0\\1\\1\end{bmatrix} = \left(e^{t}\begin{bmatrix}1&-1&-1\\-1&1&-1\\0&0&2\end{bmatrix}/2 + e^{3t}\begin{bmatrix}1&1&1\\1&1&1\\0&0&0\end{bmatrix}/2\right)\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$y = \frac{e^{t}}{2}\begin{bmatrix}-2\\0\\2\end{bmatrix} + \frac{e^{3t}}{2}\begin{bmatrix}2\\2\\0\end{bmatrix}$$

$$\boxed{y = e^{t}\begin{bmatrix}-1\\0\\1\end{bmatrix} + e^{3t}\begin{bmatrix}1\\1\\0\end{bmatrix}}$$

check: $\begin{bmatrix}1&-1&-1\\-1&1&-1\\0&0&2\end{bmatrix}/2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ $\quad \begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

eigenvector w/ value 1

$\begin{bmatrix}1&1&1\\1&1&1\\0&0&0\end{bmatrix}/2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\quad \begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\3\\0\end{bmatrix}$

eigenvector w/ value 3

so $\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\quad$ (we project $\begin{bmatrix}0\\1\\1\end{bmatrix}$ to each eigenspace)

$$\begin{array}{lll} A: & 1 & 3 \\ e^{At}: & e^{t} & e^{3t} \end{array}$$

$$\Rightarrow \quad y = e^{t}\begin{bmatrix}-1\\0\\1\end{bmatrix} + e^{3t}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$\boxed{3G1}$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$r+s+t = \text{trace} = 6$

$rs+rt+st = \text{huh}$

$$= \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 12 \quad - 4 \quad + 4 \quad = 12$$

$$rst = \det(A) = 2\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 2(4+6-6) = 8$$

$$-\det|A-\lambda I| = \lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

-2	$-8 \quad -24 \quad -24 \quad -8$
2	$\boxed{8 \quad -24 \quad +24 \quad -8 \quad = 0}$ ✓
-1	$-1 \quad -6 \quad -12 \quad -8$
1	$1 \quad -6 \quad +12 \quad -8 \quad = -1 \neq 0$
-4	very negative
4	$64 \quad -96 \quad +48 \quad -8 \quad = 8 \neq 0$
-8	very negative
8	but we already found 2 dividing det = 8

Huh.

repeated roots are roots of derivative:

$$f(x) = (x-a)^2 g(x) \Rightarrow f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x)$$

$$\Rightarrow f'(a) = 0$$

$$(\lambda^3 - 6\lambda^2 + 12\lambda - 8)' = 3\lambda^2 - 12\lambda + 12 \qquad 3\cdot 4 - 12\cdot 2 + 12 = 0 \text{ ✓}$$

$$(3\lambda^2 - 12\lambda + 12)' = 6\lambda - 12 \qquad 6\cdot 2 - 12 = 0 \text{ ✓}$$

$\lambda = 2,2,2$

$(\lambda-2)^3 = \lambda^3 - 6\lambda^2 + 12\lambda - 8$ ✓

Let $B = A - 2I \qquad B^3 = (A-2I)^3 = 0$

$$A = 2I + B$$

$$A^n = 2^n I + n2^{n-1}B + \frac{n(n-1)}{2}2^{n-2}B^2$$

$$B = \begin{bmatrix} 0 & 3 & 3 \\ -2 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix} \qquad B^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & -6 \\ 0 & 6 & 6 \end{bmatrix}$$

...3G1

$$\boxed{A^n = 2^n\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + n2^{n-1}\begin{bmatrix}0&3&3\\-2&1&1\\2&-1&-1\end{bmatrix} + \binom{n}{2}2^{n-2}\begin{bmatrix}0&0&0\\0&-6&-6\\0&6&6\end{bmatrix}}$$

check: we can see $n=0,1$ by construction

$$n=2: \quad \begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix} + \begin{bmatrix}0&12&12\\-8&4&4\\8&-4&-4\end{bmatrix} + \begin{bmatrix}0&0&0\\0&-6&-6\\0&6&6\end{bmatrix} = \begin{bmatrix}4&12&12\\-8&2&-2\\8&2&6\end{bmatrix}$$

$$\begin{bmatrix}2&3&3\\-2&3&1\\2&-1&1\end{bmatrix}^2 = \begin{bmatrix}4&12&12\\-8&2&-2\\8&2&6\end{bmatrix} \quad ✓$$

---

easier approach once we find one $\lambda=2$:

$$\left.\begin{aligned} r+s+t&=6\\ rs+rt+st&=12\\ rst&=8\end{aligned}\right\} \quad \underset{(**)}{\overset{r=2}{\Longrightarrow}} \quad \begin{aligned}2+s+t&=6\\2st&=8\end{aligned} \quad \Rightarrow \quad \begin{aligned}s+t&=4\\st&=4\end{aligned} \quad \Rightarrow \quad s,t=2,2$$

---

this is long division in disguise

$$\begin{array}{r|l} & \lambda^2-4\lambda+4 \\ \hline \lambda-2 & \lambda^3-6\lambda^2+12\lambda-8 \\ & \lambda^3-2\lambda^2 \\ \hline & \quad -4\lambda^2+12\lambda-8 \\ & \quad -4\lambda^2+8\lambda \\ \hline & \qquad\qquad 4\lambda-8 \\ & \qquad\qquad 4\lambda-8 \\ \hline & \qquad\qquad\quad 0 \end{array} \quad \xrightarrow{\text{faster}} \quad \begin{array}{|cc|l} 1 & 1 & \\ -6 & -4 & \text{quotient} \\ 12 & 4 & \\ -8 & \boxed{0} & \text{remainder} \end{array}$$

2·above + left

$$-6\overline{)\overset{1}{-4}} = -6+2\cdot 1 \qquad 12\overline{)\overset{-4}{4}} = 12+2(-4) \ \ldots$$

or

$$\begin{array}{r|cc|l} & 1 & 1 & \\ r+s+t & 6 & 4 & s+t \\ rs+rt+st & 12 & 4 & st \\ rst & 8 & \boxed{0} & \text{(it worked)} \end{array}$$

−2·above + left
−r·above + left

don't bother with ± signs of polynomial, just automate removing $r=2$

But (**) above is easier to recall.

3H1

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}$$

$r+s+t = 6$

$rs+rt+st = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}$

$= -1 + 9 + 4 = 12$

$rst = 2\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$

$= 2\cdot 4 - 3\cdot 2 + 3\cdot 2 = 8$

Looks familiar.
lets remember triple roots, not that many.

(if $r=s=t$)

r	$3r$	$3r^2$	r^3
-2	-6	12	-8
-1	-3	3	-1
0	0	0	0
1	3	3	1
2	6	12	8
3	9	27	27

alt signs

$\lambda = 2, 2, 2$

---

so $A = 2I + N$, $N^3 = 0$ (because $(A-2I)^3 = 0$)

$$N = \begin{bmatrix} 0 & 3 & 3 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix} \qquad N^3 = 0 \checkmark$$

$$\begin{aligned} e^{At} &= e^{2It} e^{Nt} \\ &= e^{2t} I \left(I + Nt + N^2 t^2/2 \right) \\ &= e^{2t} I + te^{2t} N + t^2 e^{2t}/2 \, N^2 \end{aligned}$$

$$\boxed{e^{At} = e^{2t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 0 & 3 & 3 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} + \frac{t^2 e^{2t}}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix}}$$

$\boxed{\text{III } 3H1}$

check:

$$A^n = 2^n \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + n2^{n-1} \begin{bmatrix} 0 & 3 & 3 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} + \binom{n}{2} 2^{n-2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix}$$

$n=0,1$ by construction

$n=2$:

4	12	12
4	4-4+3	-4+3
-4	4-3	4+4-3

$$= \begin{bmatrix} 4 & 12 & 12 \\ 4 & 3 & -1 \\ -4 & 1 & +5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}^2 \checkmark$$

$\boxed{3I1}$

$$A = \begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$\text{trace} = 3$

$$\text{huh} = \begin{vmatrix} -1 & 3 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= 1 + (-1) + 3 = 3$$

$$\det = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}$$
$$= -3 + 5 - 1 = 1$$

$\boxed{\underbrace{1,1,1}_{\text{sum}=3}}$ or $\underbrace{1,-1,-1}_{\text{sum}=-1}$ has product 1

$\lambda = 1, 1, 1$

$$N = A - 1I = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad N^3 = 0 \checkmark$$

$$e^{At} = e^t \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + te^t \begin{bmatrix} -2 & 3 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \frac{t^2 e^t}{2} \begin{bmatrix} 0 & -2 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$e^{At} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \boxed{y = e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2 e^t}{2} \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}}$$

check:

$$y' = e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (te^t + e^t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{2te^t + t^2 e^t}{2}\right) \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$
$$= e^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \frac{t^2 e^t}{2} \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

and $$Ay = e^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + \frac{t^2 e^t}{2} \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} \checkmark$$

3J1 $\quad A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ symmetric, $\perp$ eigenvectors

trace = 5

we can see two eigenvectors by inspection:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$ $\lambda = 3$, rows all add up to 3

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0,$$ $\lambda = 0$, matrix is singular

(go with what you notice!)

$\lambda + 3 + 0 = 5 \Rightarrow$ remaining $\lambda = 2$

$$\lambda = 2: \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0 \checkmark$$

$\lambda = 0$

$$\begin{array}{c|ccc} & 1 & -2 & 1 \\ \hline 1 & 1 & -2 & 1 \\ -2 & -2 & 4 & -2 \\ 1 & 1 & -2 & 1 \end{array} /6$$

$\lambda = 2$

$$\begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{array} /2$$

$\lambda = 3$

$$\begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} /3$$

$$\boxed{A^n = \frac{0^n}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} + \frac{2^n}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{3^n}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

check: $\quad n=0$: $$\begin{array}{c|c|c} 1+3+2 & -2+2 & 1-3+2 \\ \hline -2+2 & 4+2 & -2+2 \\ \hline 1-3+2 & -2+2 & 1+3+2 \end{array} /6 = I \checkmark$$

$n=1$: $$\begin{array}{c|c|c} 1+1 & 1 & -1+1 \\ \hline 1 & 1 & 1 \\ \hline -1+1 & 1 & 1+1 \end{array} = A \checkmark$$

$\boxed{3K1}$

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -3 \end{bmatrix} \qquad \text{symmetric matrix}$$

First and third columns look similar.

$$\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} \qquad \text{Huh.}$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} \quad \text{so eigenvalue } -3$$

Find instead eigenvalues of $A+3I$ singular and shift back.

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \begin{aligned} \text{trace} &= 1 = 0 + s + t \\ \text{huh} &= \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1 + 0 - 1 = -2 \\ &= 0s + 0t + st \end{aligned}$$

$$s + t = 1, \quad st = -2 \Rightarrow -1, 2$$

$$B + I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

$$B - 2I \quad \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 0$$

λ for B	λ for A	vector
0	$\Rightarrow -3$	$(1,0,1)$
-1	-4	$(1,-1,-1)$
2	-1	$(1,2,-1)$

Does this really work?

$$Av = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \checkmark$$

$$Av = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \checkmark$$

$$\lambda = -4 \qquad\qquad \lambda = -3 \qquad\qquad \lambda = -1$$

$$\begin{array}{c|ccc} & 1 & -1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array} \Big/ 3 \qquad \begin{array}{c|ccc} & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array} \Big/ 2 \qquad \begin{array}{c|ccc} & 1 & 2 & -1 \\ \hline 1 & 1 & 2 & -1 \\ 2 & 2 & 4 & -2 \\ -1 & -1 & -2 & 1 \end{array} \Big/ 6$$

...3K1

So

$$\boxed{A^n = \frac{(-4)^n}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} + \frac{(-3)^n}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{(-1)^n}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}}$$

check

$n=0$ I ✓

2+3+1	-2+2	-2+3-1
-2+2	2+4	2-2
-2+3-1	2-2	2+3+1

/6

$n=1$ A

-8-9-1	8-2	8-9+1
8-2	-8-4	-8+2
8-9+1	-8+2	-8-9-1

/6

$$= \begin{bmatrix} -18 & 6 & 0 \\ 6 & -12 & -6 \\ 0 & -6 & -18 \end{bmatrix} / 6$$

✓

$\boxed{3L1}$ $2x^2+2y^2+2xz+2yz+z^2$

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & 2 & & 1 \\ y & & 2 & 1 \\ z & 1 & 1 & 1 \end{array} \qquad A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

trace $= 2+2+1 = 5$

huh $= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 4+1+1 = 6$

det $= 0$, matrix is visibly singular, 0 is an eigenvalue

$$\left.\begin{array}{l} 0+s+t=5 \\ 0s+0t+st=6 \end{array}\right\} 2,3 \qquad \lambda = 0,2,3$$

$A = 0B_0 + 2B_2 + 3B_3$, B_0 drops out

B_2: $\lambda=2$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$ $\quad \begin{array}{c|ccc} & 1 & -1 & 0 \\ \hline 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} /2$

B_3: $\lambda=3$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ $\quad \begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} /3$

so $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ✓

$$[x\ y\ z]\,A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x\ y\ z]\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + [x\ y\ z]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= (x^2-2xy+y^2) + x^2+y^2+z^2+2xy+2xz+2yz \quad ✓$$

$$= \boxed{(x-y)^2 + (x+y+z)^2}$$

check: $x-y = (1,-1,0)\cdot(x,y,z)$ $\qquad (1,-1,0)\cdot(1,1,1) = 0$ $\perp$ ✓

$x+y+z = (1,1,1)\cdot(x,y,z)$