Practice Exam 3

Linear Algebra, Dave Bayer, November 19, 2013

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more than one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$-1\begin{vmatrix}2&1&1\\1&2&2\\0&1&1\end{vmatrix} + 2\begin{vmatrix}2&1&2\\1&2&1\\0&1&2\end{vmatrix} = 2\left(-1\begin{vmatrix}2&2\\1&1\end{vmatrix} + 2\overbrace{\begin{vmatrix}2&1\\1&2\end{vmatrix}}^{3}\right)$$

same, det=0

same, det=0

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= 2\cdot 2(4-1) = \boxed{12}$$

check:

$$\begin{vmatrix}2&1&2&1\\1&2&1&2\\0&1&2&1\\0&0&1&2\end{vmatrix} = -1\underbrace{\begin{vmatrix}1&1&2\\0&2&1\\0&1&2\end{vmatrix}}_{3} + 2\underbrace{\begin{vmatrix}2&2&1\\0&2&1\\0&1&2\end{vmatrix}}_{6} - 1\begin{vmatrix}2&2&1\\1&1&2\\0&1&2\end{vmatrix}$$

$$\Downarrow \quad ② \leftarrow ② - ③$$

$$\underbrace{\begin{vmatrix}2&2&1\\1&0&0\\0&1&2\end{vmatrix}}_{-3}$$

$$= -3 + 12 + 3 = \boxed{12}$$

check:

$$\begin{bmatrix}2&1&2&1\\1&2&1&2\\0&1&2&1\\0&0&1&2\end{bmatrix} \Longrightarrow \begin{bmatrix}2&0&0&0\\1&2&0&0\\0&1&2&1\\0&0&1&2\end{bmatrix} = \begin{vmatrix}2&0\\1&2\end{vmatrix}\cdot\begin{vmatrix}2&1\\1&2\end{vmatrix} = 4\cdot 3$$

$① \leftarrow ① - ③$
$② \leftarrow ② - ④$
row ops

block triangular

$$= \boxed{12}$$

[2] Find the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$f(0) = 1$

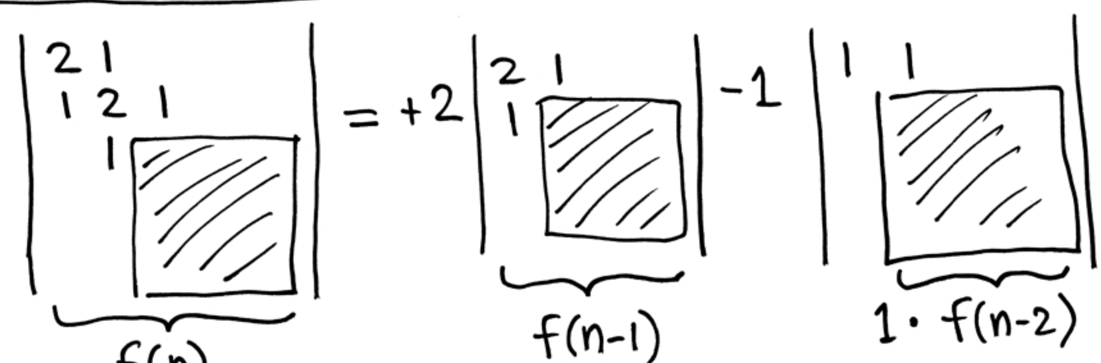
$f(1) = |2| = 2$

$f(2) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$f(3) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\underbrace{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}_{3} - 1\underbrace{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}}_{2} = 4$

$\ldots f(7) = \boxed{8}$

check



$f(n) = 2f(n-1) - f(n-2)$

n	$f(n)$
0	1
1	2
2	3
3	4
4	5
5	6
6	7
7	8

$\boxed{\det = 8}$

check:

$f(2) = af(1) + bf(0)$

$f(3) = af(2) + bf(1)$

$3 = 2a + b$

$4 = 3a + 2b$

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ✓

$f(n) = 2f(n-1) - f(n-2)$
is pattern to $f(0), f(1), f(2), f(3)$

[3] Find w/y where

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Cramer's rule

$$w = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix} \Big/ \det(A) \qquad y = \begin{vmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} \Big/ \det(A)$$

sum, so det=0

block triangular

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \neq 0$$

$$w/y = 0/\text{nz} = \boxed{0}$$

---

check

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

row reduce simultaneously

$\underbrace{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}}_{\text{new}} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \underbrace{\begin{matrix} 1 - 2 \\ 2 - 3 \\ 3 - 4 \end{matrix}}_{\text{old versions}}$

so $w=0$, $y=1$

$$\boxed{w/y = 0/1 = 0}$$

[4] Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{c|cccc} 2 & 1 & 1 & 2 & 1 \\ \hline 1 & 2 & 1 & 1 & 2 \\ 3 & 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \end{array}$$

$$\boxed{\begin{bmatrix} 3 & 1 & -5 \\ -1 & 1 & 1 \\ -1 & -1 & 3 \end{bmatrix} / 2}$$

check product = I ✓

---

check

$[A \mid I] \Downarrow [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Downarrow$ 1 ← 1 − 3, 2 ← 2 − 3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Downarrow$ 3 ← 3 − 1 − 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 & -1 & 3 \end{array}\right]$$

$\Downarrow$ 3 ← 3/2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] / 2$$

(double other rows so whole matrix is/2)

$\Downarrow$ 1 ← 1 − 3, 2 ← 2 + 3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] / 2$$

$$\boxed{\begin{bmatrix} 3 & 1 & -5 \\ -1 & 1 & 1 \\ -1 & -1 & 3 \end{bmatrix} / 2}$$

[5] Find A^n where A is the matrix

$$\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\det\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda I\right| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc)$$

$$\det\left|\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \lambda I\right| = \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-2)(\lambda-3) = 0 \qquad \lambda = 2, 3$$

$$\lambda = 2 \quad A - 2I: \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\lambda = 3 \quad A - 3I: \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0 \qquad A = CDC^{-1}$$

$$A = \underset{S \leftarrow S}{\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}} = \underset{S \leftarrow V}{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}} \underset{V \leftarrow V}{\begin{bmatrix} 2 & \\ & 3 \end{bmatrix}} \underset{V \leftarrow S}{\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}} \Big/ 1$$

could do in head

$$= 2\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\Big/1 + 3\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\Big/1$$

$$= 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \end{bmatrix}\Big/1 + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\Big/1$$

$$\boxed{A^n = 2^n\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + 3^n\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}}$$

check

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \quad ✓$$

$$\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = 2\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + 3\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \quad ✓$$

[5] Find A^n where A is the matrix

$$\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\text{trace} = \lambda_1 + \lambda_2 = 4+1 = 5$$
$$\det = \lambda_1 \lambda_2 = \begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} = 6 \quad \Rightarrow \quad \lambda = 2, 3$$

$$\lambda = 2 \quad A - 2I: \overbrace{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}}^{0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} / 1 = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

$$\lambda = 3 \quad A - 3I: \overbrace{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}}^{0} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} / 1 = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\boxed{A^n = 2^n \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + 3^n \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}}$$

---

check: $n=0 \quad I = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ ✓

$n=1 \quad A = 2\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + 3\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -2 & -4 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ ✓

[5] Find A^n where A is the matrix

$$\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\text{sum} = 4+1 = 5 \qquad \Rightarrow \lambda = 2,3$$
$$\text{prod} = \begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} = 6$$

isolate 2 isolate 3

$$\left.\begin{aligned} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} &= 2\begin{bmatrix} \; \end{bmatrix} + 3\begin{bmatrix} \; \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} &= 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}\right\} \text{same rule in any coordinate system}$$

$$x = 2\left[\frac{x-3}{2-3}\right] + 3\left[\frac{x-2}{3-2}\right]$$

x	$\frac{x-3}{2-3}$	$\frac{x-2}{3-2}$
2	1	0
3	0	1
	isolate 2	isolate 3

$$\frac{A-3}{2-3} = -\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

$$\frac{A-2}{3-2} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\boxed{A^n = 2^n \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + 3^n \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}}$$

---

check:

$$I = 1\begin{bmatrix} \; \end{bmatrix} + 1\begin{bmatrix} \; \end{bmatrix} \quad ✓$$
$$A = 2\begin{bmatrix} \; \end{bmatrix} + 3\begin{bmatrix} \; \end{bmatrix} \quad ✓$$