Final Exam

Linear Algebra, Dave Bayer, December 19, 2013

[1] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$① \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$② \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$① \qquad \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$② \qquad \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \qquad \begin{array}{l} x = 1 \\ y - z = 0 \end{array}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t}$$

check:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$$

$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ in ① but not ②: $\quad y = z$ for ②

$\qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for ①

[2] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ defined by the equation $w + x - y - z = 0$. Extend this basis to a orthogonal basis for $\mathbb{R}^4$.

$[1\ 1\ {-1}\ {-1}]$ $\perp$ to subspace, dim 3

$$\left.\begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\right\}$$ in subspace, $\perp$ to each other

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow$$ kernel vector $(1,1,1,1)$ $\perp$ to all

$$\boxed{\begin{array}{l} \left.\begin{array}{l} v_1 = (1,-1,0,0) \\ v_2 = (0,0,1,-1) \\ v_3 = (1,1,1,1) \end{array}\right\} \perp \text{ basis, subspace} \\ \hline \left. v_4 = (1,1,-1,-1) \right\} \perp \text{ extension to } \mathbb{R}^4 \end{array}}$$

[3] Find the determinant of the matrix

$$f(7) = \operatorname{Det}\left(\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix} \right)$$

$$f(0) = 1$$
$$f(1) = 2$$
$$f(2) = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2$$

$$f(n) = 2f(n-1) - 2f(n-2)$$

$$f(3) = 2\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 0$$

$$f(4) = 2\begin{vmatrix} \boxed{\begin{matrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 2 \end{matrix}} \end{vmatrix} - 2\begin{vmatrix} 1 & 0 & 0 \\ 2 & \boxed{\begin{matrix} 2 & 1 \\ 2 & 2 \end{matrix}} \\ 0 & & \end{vmatrix} = 2f(3) - 2f(2) = -4$$

n	$f(n)$
0	1
1	2
2	2
3	0
4	-4
5	-8
6	-8
7	0

$$\boxed{\operatorname{Det} = 0}$$

Check: Is matrix singular?
Try finding kernel vector.

$$\begin{bmatrix} 1 \\ -2 \\ \\ \\ \\ \\ \\ \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -2 \\ 2 \\ \\ \\ \\ \\ \end{bmatrix} \Rightarrow \dots \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \\ -4 \\ 8 \\ -8 \end{bmatrix} \checkmark$$

[4] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1, 3 \qquad e^{At} = \frac{e^{-t}}{4}\begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} + \frac{e^{3t}}{4}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \qquad y = \frac{e^{-t}}{4}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{e^{3t}}{4}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

sum = 2
prod = −3 $\Rightarrow \lambda = -1, 3$

$A+I: \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 0 \qquad \lambda = -1 \qquad e^{-t}$

$A-3I: \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \qquad \lambda = 3 \qquad e^{3t}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{y = \frac{e^{-t}}{4}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{e^{3t}}{4}\begin{bmatrix} 3 \\ 3 \end{bmatrix}}$$

Check: $Ay = \frac{e^{-t}}{4}\underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{\begin{bmatrix} -1 \\ 3 \end{bmatrix}} + \frac{e^{3t}}{4}\underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix}}_{\begin{bmatrix} 9 \\ 9 \end{bmatrix}}$

$$y' = -\frac{e^{-t}}{4}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{3e^{3t}}{4}\begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad ✓$$

[5] Express the quadratic form

$$-4xy + 3y^2$$

as a sum of squares of othogonal linear forms.

$$\lambda = -1, 4 \qquad A = \begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} + \frac{4}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$-4xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{5}(2x+y)^2 + \frac{4}{5}(x-2y)^2$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{matrix} \text{sum} = 3 \\ \text{prod} = -4 \end{matrix} \Rightarrow \lambda = -1, 4$$

$$\lambda = -1 \quad A+I: \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0 \qquad \begin{array}{c|cc} & 2 & 1 \\ \hline 2 & 4 & 2 \\ 1 & 2 & 1 \end{array} /5$$

$$\lambda = 4 \quad A-4I: \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0 \qquad \begin{array}{c|cc} & 1 & -2 \\ \hline 1 & 1 & -2 \\ -2 & -2 & 4 \end{array} /5$$

$$\Rightarrow A = -\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}/5 + 4\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}/5 \qquad \begin{matrix} n=0 \checkmark \\ n=1 \checkmark \end{matrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = -\left(\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)^2/5 + 4\left(\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)^2/5$$

$$= \boxed{-\frac{1}{5}(2x+y)^2 + \frac{4}{5}(x-2y)^2}$$

check:

	x^2	xy	y^2
$-1/5$	4	4	1
$4/5$	1	-4	4
	0	-4	3

$\checkmark$

[6] Solve the recurrence relation

$$f(0) = a, \quad f(1) = b, \quad f(n) = 3\,f(n-1) \;-\; 2\,f(n-2)$$

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} + 2^n \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$$

$$f(n) = (-b + 2a) + 2^n (b - a)$$

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \qquad \begin{array}{c} \quad\;\; 3 \quad\; -2 \\ f(n) - 3f(n-1) + 2f(n-2) = 0 \end{array}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0 \qquad \lambda = 1, 2$$

$$\lambda = 1: \quad A - I \qquad \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{array}{c|cc} & 1 & -2 \\ \hline 1 & 1 & -2 \\ 1 & 1 & -2 \end{array} \Big/ -1$$

$$\lambda = 2: \quad A - 2I \qquad \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{array}{c|cc} & 1 & -1 \\ \hline 2 & 2 & -2 \\ 1 & 1 & -1 \end{array} \Big/ 1$$

$$\Rightarrow \quad A^n = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} + 2^n \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \qquad \begin{array}{l} n=0 \;\checkmark \\ n=1 \;\checkmark \end{array}$$

$$\begin{bmatrix} \equiv \\ f(n) \end{bmatrix} = A^n \begin{bmatrix} b \\ a \end{bmatrix} = \underbrace{\boxed{(-b + 2a) + 2^n(b-a) = f(n)}}$$

check:

n	$f(n)$
0	a
1	b
2	$3b - 2a$
3	$7b - 6a$

$(2^n - 1)\,b$	$+ (2 - 2^n)\,a$	n
3	-2	2
7	-6	3

$\checkmark$

[7] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\lambda = 0, 2, 2 \qquad e^{At} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

A singular $\Rightarrow \lambda = 0$ trace$(A) = 4 = 0 + s + t$

$$\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = 4 = 0s + 0t + st$$

2 0 2

sum, prod 4

2, 2

$\lambda = 0, 2, 2$

$\lambda = 0$ $\quad [1\ 0\ {-1}]\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad \begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{array} /2$ isolates $A = 0$

$\lambda = 2, 2$ $\quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}/2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}/2$ isolates $(A-2)^2 = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}/2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A^n = 0^n \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}/2 + 2^n \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}/2 + n2^{n-1}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$n = 0$ ✓

$n = 1$ ✓

$$\boxed{e^{At} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}}$$

[8] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} -2 & 2 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 0, 0, 0$$

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} -2 & 2 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 3 & -3 & -3 \\ 3 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

A singular $\Rightarrow \lambda = 0$ $\quad$ trace(A) = 0

$$\underbrace{\begin{vmatrix} -2 & 2 \\ -1 & 1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} -2 & -1 \\ -1 & 1 \end{vmatrix}}_{-3} + \underbrace{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}_{3} = 0$$

$\Rightarrow A^3 = 0$

$\lambda = 0, 0, 0$

$$e^{At} = I + At + \tfrac{1}{2}A^2t^2$$

$$Ay(0) = \begin{bmatrix} -2 & 2 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix}$$

$$\boxed{y = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}}$$

$$A^2 = \begin{bmatrix} 3 & -3 & -3 \\ 3 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

check: $\quad y' = \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

$Ay = \quad \checkmark \quad \checkmark \quad + \frac{t^2}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

see above $\quad$ see above

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \xmapsto{A} \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix} \xmapsto{A} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \xmapsto{A} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$