Exam 3

Linear Algebra, Dave Bayer, November 19, 2013

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more than one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 2 & 3 & 3 & 2 \\ 1 & 4 & 2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 3 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 2 & 3 & 3 & 2 \\ 1 & 4 & 2 & 3 \end{vmatrix} \xrightarrow[4 \leftarrow 4 - 2]{3 \leftarrow 3 - 1} \begin{vmatrix} \boxed{\begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix}} & \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 5 \cdot 3 = \boxed{15}$$

block triangular

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check : expand row 1

$$+2 \begin{vmatrix} 4 & 1 & 1 \\ 3 & 3 & 2 \\ 4 & 2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 4 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 3 \\ 1 & 4 & 2 \end{vmatrix}$$

$$\Downarrow 3 \leftarrow 3 - 1 \qquad \Downarrow \begin{matrix} 2 \leftarrow 2 - 2\,1 \\ 3 \leftarrow 3 - 1 \end{matrix} \qquad \Downarrow \begin{matrix} 2 \leftarrow 2 - 2\,1 \\ 3 \leftarrow 3 - 1 \end{matrix} \qquad \Downarrow \begin{matrix} 2 \leftarrow 2 - 2\,1 \\ 3 \leftarrow 3 - 1 \end{matrix}$$

$$\begin{vmatrix} 4 & 1 & 1 \\ 3 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix} \qquad \begin{vmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{\begin{matrix} 1 & 0 \\ 1 & 2 \end{matrix}} \\ 0 & \end{vmatrix} \qquad \begin{vmatrix} 1 & 4 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{vmatrix} \qquad \begin{vmatrix} 1 & 4 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$4 \cdot 4 - 3 \cdot 1 \qquad\qquad\qquad\qquad$$

$$13 \qquad 2 \qquad -10 \qquad -5$$

$$2(13) - 3(2) + 1(-10) - 1(-5) = 26 - 6 - 10 + 5 = \boxed{15}$$

[2] Find the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad n = 7$$

$f(0) = 1$

$f(1) = 2$

$f(2) = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$

$f(3) = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = -4$

$$f(4) = \begin{vmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2\underbrace{\begin{vmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix}}_{f(3)} - 3\begin{vmatrix} 1 & 3 & \\ 0 & \boxed{\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}} \\ 0 & \underbrace{}_{f(2)} \end{vmatrix} = 2f(3) - 3f(2)$$

$f(n) = 2f(n-1) - 3f(n-2)$

n	0	1	2	3	4	5	6	7
$f(n)$	1	2	1	-4	-11	-10	13	56

$2(1) - 3(2) = -4$
$2(-4) - 3(1) = -11$
$2(-11) - 3(-4) = -10$
$2(-10) - 3(-11) = 13$
$2(13) - 3(-10) = 56$

$\boxed{56}$

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check:

$1 = 2 \cdot 2 - 3 \cdot 1$ ✓
$-4 = 2 \cdot 1 - 3 \cdot 2$ ✓

$$\begin{bmatrix} f(3) \\ f(2) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} f(2) \\ f(1) \end{bmatrix}$$

$$\ldots \begin{bmatrix} f(7) \\ f(6) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}^5 \begin{bmatrix} f(2) \\ f(1) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -6 \\ 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & -6 \\ 2 & -3 \end{bmatrix}^2 = \begin{bmatrix} -11 & 12 \\ -4 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -11 & 12 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -10 & 33 \\ & \end{bmatrix}$$

$$\begin{bmatrix} -10 & 33 \\ & \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 56 \\ \ \end{bmatrix}$$ ✓

[3] Find x/y where

$$\begin{bmatrix} a & b & c & d \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$x/y = \frac{\begin{vmatrix} a & a & c & d \\ 0 & b & 1 & 1 \\ 0 & c & 1 & 1 \\ 0 & d & 0 & 1 \end{vmatrix} / D}{\begin{vmatrix} a & b & a & d \\ 0 & 1 & b & 1 \\ 0 & 0 & c & 1 \\ 0 & 0 & d & 1 \end{vmatrix} / D}$$

$$a\begin{vmatrix} b & 1 & 1 \\ c & 1 & 1 \\ d & 0 & 1 \end{vmatrix} = a\begin{vmatrix} b & 1 \\ c & 1 \end{vmatrix} = a(b-c)$$

$$a\begin{vmatrix} 1 & b & 1 \\ 0 & c & 1 \\ 0 & d & 1 \end{vmatrix} = a\begin{vmatrix} c & 1 \\ d & 1 \end{vmatrix} = a(c-d)$$

$$\boxed{x/y = \frac{b-c}{c-d}}$$

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check: try $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow x/y = \frac{b-c}{c-d} = \frac{3-2}{2-1} = 1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} x=1 \\ y=1 \end{matrix} \Rightarrow x/y = 1 \ ✓$$

subtract each row from prev

[4] Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$\begin{array}{c|cccc} & 2 & 1 & 1 & 2 \; 1 \\ \hline 1 & 2 & 3 & 1 & 2 \\ 3 & 0 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

$$\boxed{\begin{bmatrix} 0 & 9 & -6 \\ 0 & -3 & 3 \\ 1 & -5 & 3 \end{bmatrix} \Big/ 3}$$

check $A \cdot A^{-1} = I$ ✓

Check: $[A \mid I] \Rightarrow [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\Downarrow$ ① ← ① − 2② ; ③ ← ③ − ②

$$\left[\begin{array}{ccc|ccc} 0 & -3 & 3 & 1 & -2 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$\Downarrow$ ① ← ① + 3③ ; ② ← ② − 2③

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 3 & 1 & -5 & 3 \\ 1 & 0 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$\Downarrow$ first row to bottom

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & -5 & 3 \end{array}\right]$$

$\Downarrow$ ③ ← ③/3
mult other rows by 3
to put over common denominator

$$\left[\begin{array}{ccc|ccc} 1 & & & 0 & 9 & -6 \\ & 1 & & 0 & -3 & 3 \\ & & 1 & 1 & -5 & 3 \end{array}\right] \Big/ 3$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 9 & -6 \\ 0 & -3 & 3 \\ 1 & -5 & 3 \end{bmatrix} \Big/ 3}$$

✓

[5] Find A^n where A is the matrix

$$\begin{bmatrix} 2 & -1 \\ -4 & -1 \end{bmatrix}$$

$$\text{sum} = 2+(-1) = 1$$
$$\text{prod} = 2(-1) - (-1)(-4) = -6 \quad \Rightarrow \quad \lambda = -2, 3$$

$$\lambda = -2 \quad A+2I: \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}4 & -1 \\ -4 & 1\end{bmatrix}\begin{bmatrix}1 \\ 4\end{bmatrix} \qquad \begin{bmatrix}1 \\ 4\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}/5 = \begin{bmatrix}1 & 1 \\ 4 & 4\end{bmatrix}/5$$

$$\lambda = 3 \quad A-3I: \begin{bmatrix}4 & -1\end{bmatrix}\begin{bmatrix}-1 & -1 \\ -4 & -4\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix} \qquad \begin{bmatrix}1 \\ -1\end{bmatrix}\begin{bmatrix}4 & -1\end{bmatrix}/5 = \begin{bmatrix}4 & -1 \\ -4 & 1\end{bmatrix}/5$$

$$\boxed{A^n = (-2)^n \begin{bmatrix}1 & 1 \\ 4 & 4\end{bmatrix}/5 + 3^n \begin{bmatrix}4 & -1 \\ -4 & 1\end{bmatrix}/5}$$

check

$$\begin{bmatrix}1 & 1 \\ 4 & 4\end{bmatrix}/5 + \begin{bmatrix}4 & -1 \\ -4 & 1\end{bmatrix}/5 = \begin{bmatrix}5 & 0 \\ 0 & 5\end{bmatrix}/5 = I \checkmark$$

$$-2\begin{bmatrix}1 & 1 \\ 4 & 4\end{bmatrix}/5 + 3\begin{bmatrix}4 & -1 \\ -4 & 1\end{bmatrix}/5 = \begin{bmatrix}10 & -5 \\ -20 & -5\end{bmatrix}/5 = A \checkmark$$