Exam 1

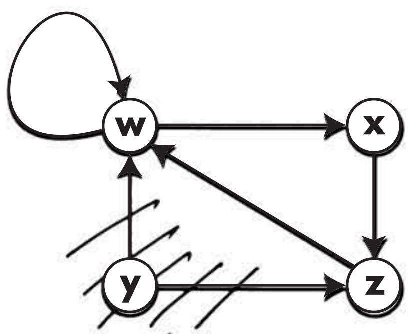
Linear Algebra, Dave Bayer, September 24, 2013

Name: __________________________________ **Uni:** ________________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Using matrix multiplication, count the number of paths of length ten from w to itself.



not used

from

$$\text{to}\ \begin{array}{c|c} & \begin{matrix} w & x & z \end{matrix} \\ \begin{matrix} w \\ x \\ z \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} = A$$

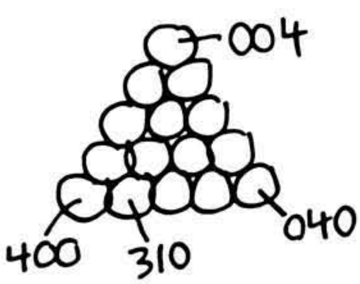
$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad A^5 = \begin{bmatrix} 4 & 2 & 3 \\ 3 & & \\ 2 & & \end{bmatrix}$$

not needed

$$A^{10} = \begin{bmatrix} 28 & & \\ & & \\ & & \end{bmatrix} \qquad 28 = 4\cdot4 + 2\cdot3 + 3\cdot2 = 16 + 6 + 6 \qquad \boxed{28 \text{ paths}}$$

Check: 0,1,2 or 3 wxz loops, rest w loops in buckets between wxz loops

wxz loops	w loops	buckets	# ways	
0	10	1	1	10
1	7	2	8	70, 61, 52, 43, 34, 25, 16, 07
2	4	3	15	400, 310, ..., 004
3	1	4	4	1000, 0100, 0010, 0001
			28 ✓	



example 310

$$\underbrace{w\,w\,w}_{\substack{3 \\ \text{before} \\ \text{bucket}}}\ \boxed{wxz}\ \underbrace{w}_{\substack{1 \\ \text{between} \\ \text{bucket}}}\ \boxed{wxz}\ \underbrace{w}_{\substack{0 \\ \text{after} \\ \text{bucket}}}$$

[2] Solve the following system of equations.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

particular solution: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ $\quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

homogeneous solutions: 3 conditions on 4 variables leaves 1 dimension solution space

$$[2\ 0\ 0\ 3]\begin{bmatrix} 3 \\ \\ \\ -2 \end{bmatrix} = 0 \quad [1\ 0\ 1\ 1]\begin{bmatrix} 3 \\ \\ -1 \\ -2 \end{bmatrix} = 0 \quad [1\ 1\ 2\ 1]\begin{bmatrix} 3 \\ 1 \\ -1 \\ -2 \end{bmatrix} = 0$$

$$\boxed{\text{general solution: } \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -1 \\ -2 \end{bmatrix} t}$$

check: $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 2 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -1 \\ -2 \end{bmatrix} t \right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} t = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ ✓

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix} \xrightarrow{(1) \rightleftarrows (2)} \begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix} \xrightarrow{(2) \leftarrow (2) - 6(1)} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \xrightarrow{(2) \leftarrow \frac{1}{3}(2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix} \xleftarrow{(1) \rightleftarrows (2)} \begin{bmatrix} 1 & 0 \\ 6 & 3 \end{bmatrix} \xleftarrow{(2) \leftarrow (2) + 6(1)} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \xleftarrow{(2) \leftarrow 3(2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right) I$$

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}$$

check $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}}_{\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix}$ ✓

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$1\left\{\bullet\right. \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 4 & 5 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{\bullet\ \bullet\ \bullet\ =\ 3} = 0$$

$$1+3=4$$

$$\begin{bmatrix} 3 & -2 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 5 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\text{system of equations:} \quad \begin{bmatrix} 3 & -2 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 5 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

check: $$\begin{bmatrix} 3 & -2 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 5 & 0 & 0 & -2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} s \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} s = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

set equal,

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

3 conditions, rows 1, 3 same

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} t$$

$$\Rightarrow \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} t}_{\begin{bmatrix} a \\ b \end{bmatrix}} \Bigg) = \boxed{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} t = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}}$$

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check:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t}_{\begin{bmatrix} c \\ d \end{bmatrix}} \Bigg) = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} t \quad \checkmark$$

(t parametrizes same points in both spaces, in lock step)