Solutions to first midterm

Dave Bayer, Modern Algebra, Exam date October 14, 1998

[1] Let $G = S_3 = \{\,(), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\,\}$ be the symmetric group of all permutations of $\{\,1, 2, 3\,\}$, and let H be the subgroup $H = \{\,(), (1\ 2)\,\} \subset G$.
(a) List the left cosets of H in G.

Solution: $H = \{\,(), (1\ 2)\,\}$ is one left coset. We expect a total of 3 left cosets, because the left cosets partition the 6 elements of G into 3 subsets of 2 elements each. The other left cosets are of the form gH for $g \in G$; we know that $g = (\ )$ and $g = (1\ 2)$ yield H itself, so we try another choice for g.

Taking $g = (1\ 3)$ we have $gH = \{\,(1\ 3), (1\ 3\ 2)\,\}$. (Calculation, working left to right to compute $(1\ 3)\,(1\ 2)$: 1 goes to 3 stays 3, 3 goes to 1 goes to 2, 2 stays 2 goes to 1, so $(1\ 3)\,(1\ 2) = (1\ 3\ 2)$.)

There are two elements of G unaccounted for, so the remaining left coset must be $\{\,(2\ 3), (1\ 2\ 3)\,\}$. Thus, the answer is

$$\{\,(), (1\ 2)\,\}, \quad \{\,(1\ 3), (1\ 3\ 2)\,\}, \quad \{\,(2\ 3), (1\ 2\ 3)\,\}.$$

(b) List the right cosets of H in G.

Solution: $H = \{\,(), (1\ 2)\,\}$ is one right coset; the other right cosets are of the form Hg. We have $H\,(1\ 3) = \{\,(1\ 3), (1\ 2\ 3)\,\}$, because $(1\ 2)\,(1\ 3) = (1\ 2\ 3)$, so the answer is

$$\{\,(), (1\ 2)\,\}, \quad \{\,(1\ 3), (1\ 2\ 3)\,\}, \quad \{\,(2\ 3), (1\ 3\ 2)\,\}.$$

(c) Is H normal in G?

Solution: No, because the left and right cosets of H in G aren't the same.

[2] Let m and n be relatively prime integers, and consider the two groups $G = \mathbb{Z}_m \times \mathbb{Z}_n$ and $H = \mathbb{Z}_{mn}$.
(a) Present each of these groups in terms of generators and relations.

$$G = \langle\, a, b \mid a^m = b^n = 1,\ ab = ba \,\rangle, \quad H = \langle\, c \mid c^{mn} = 1 \,\rangle$$

(b) Find an isomorphism between G and H.

Solution: The element ab of G has order mn, as does c, so we can define an isomorphism $f : H \to G$ by the rule $f(c) = ab$.

(c) Give an example showing what happens when m and n aren't relatively prime.

Solution: Taking $m = n = 2$, every nonidentity element of G (the Klein-four group) has order 2. Thus, G cannot be isomorphic to the cyclic group H, whose generator c has order 4.

[3] Let G be the group presented in terms of generators and relations by

$$G \quad = \quad \langle\, a, b \mid a^2 = b^2 = 1,\ bab = aba \,\rangle.$$

Describe G as completely as you can. (Suggestions: How many elements does G have? List representatives for the distinct elements of G. Is G abelian or not? Draw the Cayley graph of G. Recognize G as isomorphic to a familiar group, and give an explicit isomorphism.)

Solution: Using the rules $a^2 = 1$ and $b^2 = 1$, we can simplify any word in G so it has no repeated letters. In other words, we can simplify to words which alternate $ababab\ldots$ or $bababa\ldots$. For example, $baabab = b(aa)bab = b(1)bab = bbab = (bb)ab = (1)ab = ab$. Moreover, we can change any bab to aba as a way of standardizing words.

Playing around, we can find a bab in any alternating word of length ≥ 4, and after replacing bab by aba there will always be a repeated letter we can simplify. For example, $abab$ becomes $a(bab) = a(aba) = (aa)ba = ba$, and $baba$ becomes $(bab)a = (aba)a = ab(aa) = ab$. We can also always change bab to aba. With these simplifications, G consists of the identity 1, the words a, ab, aba which start with a, and the words b and ba which start with b. Thus G has 6 elements, and representatives for these 6 elements can be chosen as follows:

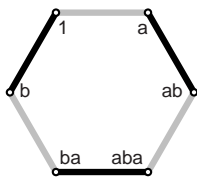$$G \quad = \quad \{\, 1,\ a,\ ab,\ aba,\ b,\ ba \,\}.$$

Figure 1

In figure 1 we draw a Cayley graph of G, where moving either way along a grey edge corresponds to right multiplication by a, and moving either way along a black edge corresponds to right multiplication by b. Starting at 1, there are two ways to reach $bab = aba$: the path bab (b then a then b) which goes counterclockwise from 1 to aba, and the path aba (a then b then a) which goes clockwise from 1 to aba.

G is not abelian, because $ab \neq ba$. There is only one nonabelian group of order 6, up to isomorphism: the group S_3 of all permutations of 1, 2, 3, which we have also encountered as the group of symmetries of a triangle. For an explicit isomorphism, identity a with $(1\ 2)$, and identify b with $(1\ 3)$. Then $a^2 = b^2 = 1$ as desired. Now check that $bab = aba$: $bab = (1\ 3)(1\ 2)(1\ 3) = (2\ 3)$, and $aba = (1\ 2)(1\ 3)(1\ 2) = (2\ 3)$.
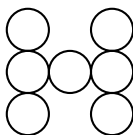

Figure 2

[**4**] Describe the group G of symmetries of the configuration of cells shown in Figure 2, considering both rotations and flips. How many ways are there of marking two of the cells in Figure 1, up to symmetry? Use Burnside's formula

$$(\# \text{ of patterns up to symmetry}) \quad = \quad \frac{1}{|G|} \sum_{g \in G} (\# \text{ of patterns fixed by } g).$$

Solution: This configuration is preserved by leaving it alone, flipping it horizontally or vertically, or rotating it a half turn. Thus G has 4 elements.

There are $\frac{7 \cdot 6}{2 \cdot 1} = 21$ ways of choosing 2 of the 7 cells, all of which are fixed by the identity.

To count patterns which are preserved by flipping across a horizontal axis, we can choose the outer left corners, the outer right corners, or any two of the three cells in the middle row. This gives $1 + 1 + 3 = 5$ possibilities.

2

To count patterns which are preserved by flipping across a vertical axis, we can choose one cell in the left column, and its counterpart in the right column under flipping. This gives 3 possibilities.

To count patterns which are preserved by rotating a half turn, we can choose one cell in the left column, and its counterpart in the right column under rotating. This gives 3 possibilities.

Thus, by Burnside's formula there are $(21 + 5 + 3 + 3)/4 = 8$ patterns, up to symmetry.
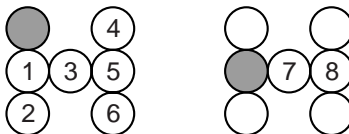

Figure 3

To check our work, we use figure 3 to confirm that there are 8 patterns up to symmetry. If any corner is marked, we can move the pattern so the upper left corner is marked, as shown on the left. Then, any of the remaining 6 cells can be marked as our second choice, and each of these patterns are different. We must also count patterns which don't mark any corner: There is only one center cell, so such patterns must mark a side cell. We can move the pattern so the left side cell is always marked, as shown on the right. There are 2 noncorner cells left for our second choice, and they give different patterns, for a total of 8.

[5] The *normalizer* $N(H)$ of a subgroup H of a group G can be defined to be the set

$$N(H) \quad = \quad \{\, g \in G \mid gHg^{-1} = H \,\}.$$

(a) Prove that $N(H)$ is a subgroup of G.

Solution: We need to check that $N(H)$ is closed under multiplication, and under taking inverses. If a, $b \in N(H)$, then $aHa^{-1} = H$ and $bHb^{-1} = H$. Then $(ab)H(ab)^{-1} = abHb^{-1}a^{-1} = aHa^{-1} = H$, so $ab \in N(H)$, and $N(H)$ is closed under multiplication. Also, multiplying $aHa^{-1} = H$ on the left by a^{-1} and on the right by a yields $H = a^{-1}Ha$, showing that $a^{-1} \in N(H)$, so $N(H)$ is closed under taking inverses.

(b) Prove that H is a normal subgroup of $N(H)$.

Solution: We already know that H is a group, because it is a subgroup of G. H is contained in $N(H)$ because $gHg^{-1} = H$ for every $g \in H$. Thus, H is a subgroup of $N(H)$. H is normal in $N(H)$ because $gHg^{-1} = H$ for every $g \in N(H)$, by the definition of $N(H)$.

(c) Suppose that J is another subgroup of G *conjugate* to H: $H \neq J$, but $aHa^{-1} = J$ for some $a \in G$. Describe the set $\{\, g \in G \mid gHg^{-1} = J \,\}$ in terms of a and $N(H)$.

Solution: Let $U = \{\, g \in G \mid gHg^{-1} = J \,\}$. Then U is the left coset $aN(H)$ of $N(H)$: If $g \in aN(H)$, then $g = ab$ for some $b \in N(H)$, so $gHg^{-1} = abHb^{-1}a^{-1} = aHa^{-1} = J$, so $g \in U$. Conversely, if $g \in U$, then $a^{-1}gHg^{-1}a = a^{-1}Ja = H$, so $a^{-1}g \in N(H)$, so $g = aa^{-1}g \in aN(H)$.

(d) How many subgroups of G are conjugate to H, counting H itself?

Solution: The subgroups J of G conjugate to H correspond 1:1 to the cosets $aN(H)$ of $N(H)$ in G, by part (c) above. The coset $N(H)$ itself counts the subgroup H itself. Thus, the number of conjugate subgroups is equal to the number of cosets, which is the index of $N(H)$ in G.