First homework problems

Dave Bayer, Modern Algebra, September 9, 1998

	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Figure 1

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	1
3	3	4	5	6	1	2
4	4	5	6	1	2	3
5	5	6	1	2	3	4
6	6	1	2	3	4	5

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	2	5	6	4
4	4	5	6	1	2	3
5	5	6	4	3	1	2
6	6	4	5	2	3	1

Figure 2

[**1**] Find all subgroups of the groups whose multiplication tables are shown in Figures 1 and 2.

[**2**] For the group on the left in Figure 1, a *quotient group* has been colored in: The multiplication rule for *grey* and *white* is independent of which elements are chosen to represent the grey and white "teams". Notice that the grey "team" is one of the subgroups you found in problem 1, now playing the role of the identity in the quotient group.

Find all quotient groups of the groups whose multiplication tables are shown in Figures 1 and 2. In every case, does a subgroup play the role of the identity in the quotient? Does every subgroup you found in problem 1 make an appearance in this problem?

[**3**] Write down the multiplication table for the group of pairs of integers under addition modulo (2,3)

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \quad = \quad \{ \ (0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,2) \ \}$$

where $(a,b)+(c,d) = (a+c \bmod 2,\ b+d \bmod 3)$. Does the pattern of its multiplication table look familiar? You may need to rearrange and relabel the rows and columns.

[**4**] Write down the multiplication table for the group of permutations on three elements

$$S_3 \quad = \quad \{ \ (\,),\ (123),\ (132),\ (12),\ (13),\ (23) \ \}\,.$$

Does the pattern of its multiplication table look familiar? You may need to rearrange and relabel the rows and columns.

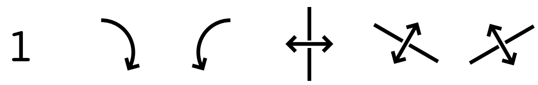

Figure 3

[**5**] There are six ways to rotate, flip, or leave alone a triangle, so each corner goes to some corner. Make up a visual notation for these six operations, such as that given in Figure 3. Write down the multiplication table for this group of symmetries of a triangle. Does the pattern of its multiplication table look familiar? You may need to rearrange and relabel the rows and columns.

[**6**] Let G be the group of all 2×2 matrices whose entries are integers mod 2, and whose determinants are nonzero. How many elements does G have? Write down the multiplication table for this group. Does the pattern of its multiplication table look familiar? You may need to rearrange and relabel the rows and columns.

[**7**] There are six ways to rotate, flip, or leave alone a triangle, so each corner goes to some corner. Make up a visual notation for these six operations. Write down the multiplication table for this group of symmetries of a square.

[**8**] Let

$$G \quad = \quad \{\, 1,\ -1,\ \mathbf{i},\ -\mathbf{i},\ \mathbf{j},\ -\mathbf{j}\ ,\ \mathbf{k},\ -\mathbf{k}\}\,.$$

be a group under multiplication $\times$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ obey the familiar cross-product rules from multivariate calculus

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \ \mathbf{k}\times\mathbf{i}=\mathbf{j},\ \ \mathbf{j}\times\mathbf{i}=-\mathbf{k},\ \ \mathbf{k}\times\mathbf{j}=-\mathbf{i},\ \ \mathbf{i}\times\mathbf{k}=-\mathbf{j}.$$

Write down the multiplication table for this group. Is this the same group as in problem 7?