# Second midterm

Dave Bayer, Modern Algebra, November 9, 1997

Please solve 5 of the following 6 problems. Each problem is worth 5 points for a total of 25 points. I will also award up to 5 bonus points (recorded separately) for particularly impressive examinations.

**[1]** Define a *vector space*. Define a *spanning set*, a *linearly independent set*, and a *basis*. Define the *dimension* of a vector space.

**[2]** Let $\mathbf{F}_{31991} = \mathbb{Z}/31991\mathbb{Z}$ be the finite field with 31991 elements.

**(a)** What is the order of the group of invertible elements of $\mathbf{F}_{31991}$?

**(b)** Find the multiplicative inverse of 2 mod 31991, any way you can. Check your answer.

**[3]** Let $V$ be a finite-dimensional vector space over a field $F$, and suppose that $v_1, \ldots, v_n$ spans $V$. Prove that $v_1, \ldots, v_n$ contains a basis of $V$.

**[4]** Let $V$ and $W$ be finite-dimensional vector spaces over a field $F$, and let $A : V \to W$ be a linear transformation. Prove that

$$\dim(V) \quad = \quad \dim(\ker A) \; + \; \dim(\operatorname{im} A).$$

**[5]** Let $W_1, \ldots, W_n$ be $n$ subspaces of a finite-dimensional vector space $V$ over a field $F$. Prove that

$$\dim(W_1 + \ldots + W_n) \quad \leq \quad \dim(W_1) \; + \; \ldots \; + \; \dim(W_n).$$

**[6]** Let $V$ be an $n$-dimensional vector space over a field $F$, and let $A : V \to V$ be a linear transformation such that $A^2 = 0$. Find a basis

$$v_1, \ldots, v_m, \quad v_{m+1}, \ldots, v_{2m}, \quad v_{2m+1}, \ldots, v_n$$

for $V$, such that

$Av_i = 0$ for $i = 1, \ldots m$,
$Av_i = v_{i-m}$ for $i = m+1, \ldots, 2m$,
$Av_i = 0$ for $i = 2m+1, \ldots n$.

Prove that $v_1, \ldots, v_n$ is a basis for $V$.