Linear Algebra, Dave Bayer

Our first exam will be held in class on Tuesday, October 11, 2016. Makeup exams will only be given under exceptional circumstances, and require a note from a doctor or a dean.

Exam 1 will consist of five questions. The following are the skills that one needs to learn for this exam:

- Use matrix multiplication to count paths in a graph.
- Find the general solution to a system of linear equations, expressed as a particular solution plus the set of all homogenous solutions.
- Given a parametrization of an affine subspace of $\mathbb{R}^n$, find a system of linear equations having this affine subspace as its general solution.
- Find a parametrization for the intersection of two affine subspaces of $\mathbb{R}^n$, given either by parametrizations or by systems of equations.
- Express a matrix as a product of elementary matrices.
- Use Gaussian elimination to find the inverse of a matrix.
- Find the matrix or the set of matrices determined by a set of conditions, such as the effect on a basis, or a description as a projection.
- Find the row space and the column space of a matrix.
- Trim a set of vectors to an independent set. Extend an independent set of vectors to a basis.
- Find a basis for a subspace of $\mathbb{R}^n$. Extend this independent set to a basis for $\mathbb{R}^n$.
- Use least squares to fit data.
- Find the orthogonal projection of a vector to a subspace of $\mathbb{R}^n$.
- Find an orthogonal basis for a subspace of $\mathbb{R}^n$. Extend this independent set to an orthogonal basis for $\mathbb{R}^n$.
- Solve familiar problems, where $\mathbb{R}^n$ is replaced by a finite dimensional space of polynomials.
- Solve familiar problems, where the dot product is replaced by an inner product given by a symmetric matrix or an integral.

This material is covered in the first five chapters of Bretscher, which you are encouraged to read carefully.

Homework will count as 10% of your course grade. The homework for Exam 1 is given below. You may turn in problems in batchs as you complete them; homework that is received early will receive more careful feedback. All homework must be submitted on or before the day of our first exam. There is a homework box on the fourth floor of the Mathematics building for your section of Linear Algebra. Please turn in homework to the box corresponding to your section. Please write your uni very clearly on each page of homework.

Please hand in the following problems, which are the same in both the 5e and 4e editions of Bretscher. (You are encouraged to work similar problems for your own practice.)

- 1.1 [10], 1.2 [10], 1.3 [24]
- 2.1 [14], 2.2 [10], 2.3 [16], 2.4 [20]
- 3.1 [24], 3.2 [32], 3.3 [20], 3.4 [24]
- 4.1 [20], 4.2 [14], 4.3 [2]
- 5.1 [16], 5.2 [14], 5.3 [36], 5.4 [22], 5.5 [10]

What follows on the remaining pages of this study guide are practice problems for our first exam, taken from past semesters of the course. The sources for the following problems, along with many solutions, can be found on our Linear Algebra Course Materials page:

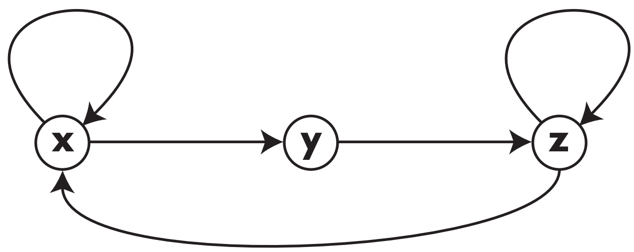
https://www.math.columbia.edu/~bayer/LinearAlgebra/

(F15 Homework 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length nine from x to z.



[3] Express A as a product of three elementary matrices, where

$$A = \begin{bmatrix} 7 & 1 \\ 4 & 0 \end{bmatrix}$$

[4] Find the matrix A such that

$$A \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

[6] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

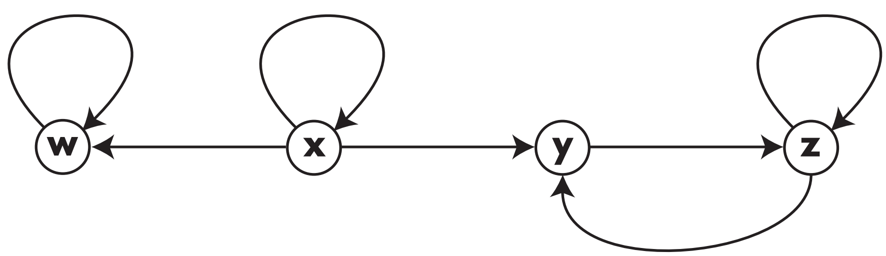
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix}$$

(F15 Exam 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length ten from x to z.



[3] Express A as a product of four elementary matrices, where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

[4] Find all 2×2 matrices A such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} u$$

(F15 Homework 2)

[1] Let v_1, v_2, $\ldots$ v_r be a set of vectors that spans a subspace W of a vector space V. Prove that one can choose a subset of these vectors that forms a basis for W, and that this basis can be extended to a basis for V. Demonstrate this procedure on the vectors

$$(1,-1,0,0) \quad (1,0,-1,0) \quad (1,0,0,-1) \quad (0,1,-1,0) \quad (0,1,0,-1) \quad (0,0,1,-1)$$

[2] Find the 3×3 matrix that vanishes on the plane $3x - 2y + z = 0$, and maps the vector $(1,0,0)$ to itself.

[3] Find the 3×3 matrix that vanishes on the vector $(1,0,2)$, and stretches each vector in the plane $x + y = 0$ by a factor of 3.

[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} t$$

[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x - 2y + z = 0$$

[6] Find the row space and the column space of the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \end{bmatrix}$$

[7] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

[8] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,2,0,0) \quad (0,1,2,0) \quad (0,0,1,2) \quad (1,0,-4,0) \quad (0,1,0,-4)$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

[9] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(0) = 0$. Find the orthogonal projection of the polynomial $x + 2$ onto the subspace W, with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$$

(F15 Exam 2) (Solutions)

[1] Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,0,1) \qquad (1,0,1,1) \qquad (2,1,1,2) \qquad (3,2,1,3) \qquad (3,1,2,3) \qquad (4,2,2,4)$$

Extend this basis to a basis for $\mathbb{R}^4$.

[2] Find the 3×3 matrix A such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

[3] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}$$

[4] Find the 4×4 matrix that projects orthogonally onto the plane spanned by the vectors $(1,0,1,0)$ and $(0,1,0,1)$.

[5] Let V be the vector space $\mathbb{R}^3$, equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle \;=\; \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Using this inner product rather than the dot product, find the orthogonal projection of the vector $(1,0,0)$ onto the plane spanned by $(0,1,0)$ and $(0,0,1)$.

(F15 Final) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

[2] Find the 3×3 matrix A such that

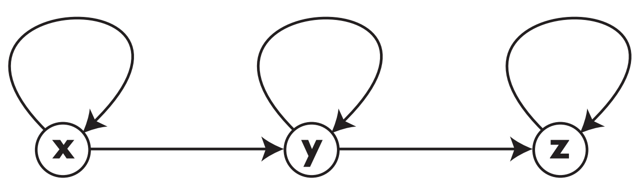
$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

(F14 Practice 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 6 & 1 & 8 & 1 \\ 4 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length ten from x to z.



[3] Express A as a product of elementary matrices, where

$$A \;=\; \begin{bmatrix} -3 & 5 \\ 1 & 0 \end{bmatrix}$$

[4] Find the matrix A such that

$$A \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \;=\; \begin{bmatrix} 3 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -3 & 0 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

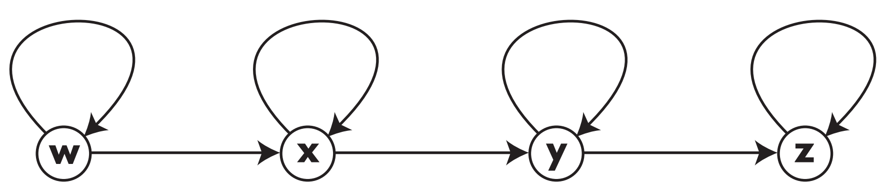
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

(F14 Homework 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 2 & -3 & -1 & 1 \\ 1 & -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \;=\; \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length ten from w to y.



[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 1 & 12 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

[4] Find the matrix A such that

$$A \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

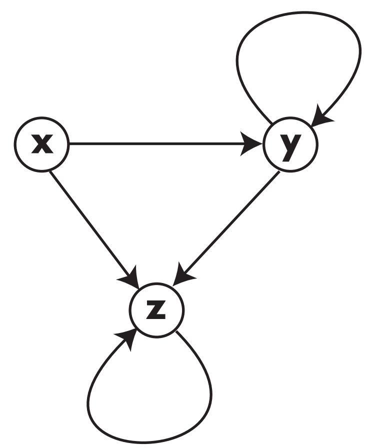
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

(F14 8:40 Exam 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length eight from x to z.



[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[4] Find the matrix A such that

$$A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

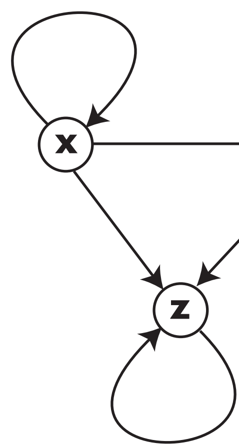
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

(F14 11:40 Exam 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length eight from x to z.



[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[4] Find the matrix A such that

$$A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

(F14 Practice 2) (Solutions)

[1] Find the 2×2 matrix which reflects across the line $x - 2y = 0$.

[2] Find the 3×3 matrix which vanishes on the plane $x + y + z = 0$, and maps the vector $(1, 0, 0)$ to itself.

[3] Find the 3×3 matrix which vanishes on the vector $(1, 1, 1)$, and maps each point on the plane $x + y = 0$ to itself.

[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} t$$

[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + y + z = 0$$

[6] Find the row space and the column space of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 \end{bmatrix}$$

[7] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

[8] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,-1,0,0) \qquad (0,1,-1,0) \qquad (0,0,1,-1) \qquad (1,0,-1,0) \qquad (0,1,0,-1)$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

[9] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(1) = 0$. Find the orthogonal projection of the polynomial x onto the subspace W, with respect to the inner product

$$\langle f, g\rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

(F14 Homework 2) (Solutions)

[1] Find the 2×2 matrix which reflects across the line $3x - y = 0$.

[2] Find the 3×3 matrix which vanishes on the plane $4x + 2y + z = 0$, and maps the vector $(1, 1, 1)$ to itself.

[3] Find the 3×3 matrix which vanishes on the vector $(1, 1, 0)$, and maps each point on the plane $x+2y+2z = 0$ to itself.

[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} t$$

[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + 2y + 3z = 0$$

[6] Find the row space and the column space of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 & 10 \end{bmatrix}$$

[7] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$$

[8] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,2,0,0) \qquad (0,1,2,0) \qquad (1,3,3,2) \qquad (0,0,1,2) \qquad (1,3,3,2)$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

[9] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(-1) = 0$. Find the orthogonal projection of the polynomial $x + 1$ onto the subspace W, with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

(F14 8:40 Exam 2) (Solutions)

[1] Find the 3×3 matrix which maps the vector $(0,1,1)$ to $(0,2,2)$, and maps each point on the plane $x+y=0$ to the zero vector.

[2] Find a basis for the row space and a basis for the column space of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 & -1 & -1 \end{bmatrix}$$

[3] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + 2y = 0$$

[4] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,-2,0,0) \qquad (1,0,-2,0) \qquad (1,0,0,-2) \qquad (0,1,-1,0) \qquad (0,1,0,-1) \qquad (0,0,1,-1)$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

[5] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of V consisting of those polynomials $f(x)$ such that the second derivative $f''(x) = 0$.
Find the orthogonal projection of the polynomial x^2 onto the subspace W, with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

(F14 11:40 Exam 2) (Solutions)

[1] Find the 3×3 matrix which maps the vector $(1,1,1)$ to $(2,2,2)$, and maps each point on the plane $x+y+z = 0$ to itself.

[2] Find a basis for the row space and a basis for the column space of the matrix

$$\begin{bmatrix} -1 & -1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -2 \\ -1 & -1 & 0 & -2 & 1 \end{bmatrix}$$

[3] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + y - 2z = 0$$

[4] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(-1, 1, 0, -1) \qquad (-1, 0, 1, -1) \qquad (0, 1, -1, 0) \qquad (-2, 1, 1, -2) \qquad (1, 1, -2, 1)$$

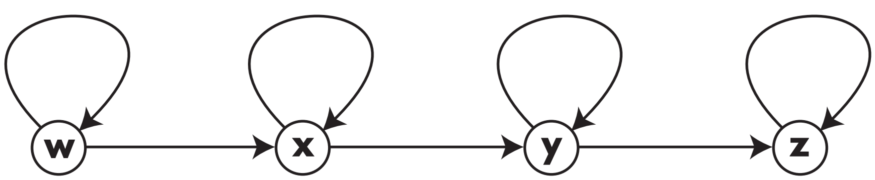
Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

[5] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of V consisting of those polynomials $f(x)$ such that the derivative $f'(0) = 0$.
Find the orthogonal projection of the polynomial x onto the subspace W, with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\, dx$$

(S14 Exam 1) (Solutions)

[1] Using matrix multiplication, count the number of paths of length six from w to z.



[2] Solve the following system of equations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 4 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

(S14 Exam 2) (Solutions)

[1] Find the row space and the column space of the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 3 & 6 & 9 & 2 \\ 0 & 4 & 8 & 2 & 6 \end{bmatrix}$$

[2] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

[3] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + 3y - 2z \ = \ 0$$

[4] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,0,0) \qquad (0,1,1,0) \qquad (0,0,1,1) \qquad (1,2,1,0) \qquad (0,1,2,1)$$

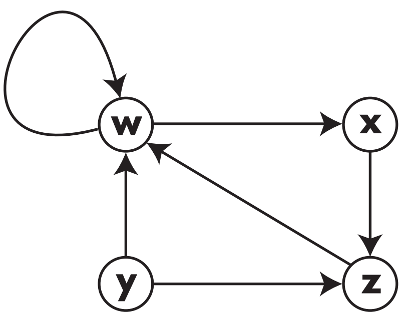
Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

[5] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials of degree $\leqslant 1$. Find the orthogonal projection of the polynomial x^2 onto the subspace W, with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\, dx$$

(F13 Exam 1) (Solutions)

[1] Using matrix multiplication, count the number of paths of length ten from w to itself.



[2] Solve the following system of equations.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \;=\; \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix}$$

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \;+\; s \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

(F13 Exam 2) (Solutions)

[1] Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(2,0,1,0),\ (2,0,0,1),\ (0,2,1,0),\ (0,2,0,1)$$

Extend this basis to a basis for $\mathbb{R}^4$.

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (-1, 0), \quad (x_2, y_2) = (0, 0), \quad (x_3, y_3) = (1, 0), \quad (x_4, y_4) = (2, 1)$$

[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the subspace V spanned by $(1, -1, 0)$ and $(0, 2, 1)$. Find the matrix A which represents L in standard coordinates.

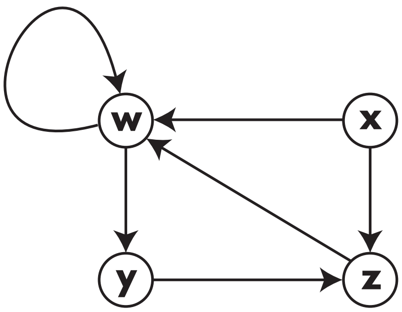
[4] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(2) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle \ = \int_0^1 f(x)g(x)\, dx$$

[5] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ defined by the equation $w + x - 2y - 2z = 0$. Extend this basis to a orthogonal basis for $\mathbb{R}^4$.

(S13 8:40 Exam 1) (Solutions)

[1] Using matrix multiplication, count the number of paths of length eight from w to itself.



[2] Solve the following system of equations.

$$\begin{bmatrix} 2 & 0 & 1 & 4 \\ 3 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} -6 & 1 \\ 3 & 0 \end{bmatrix}$$

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 3 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

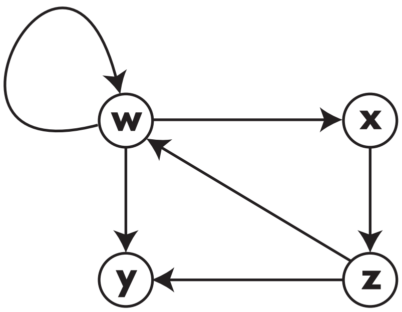
[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

(S13 10:10 Exam 1) (Solutions)

[1] Using matrix multiplication, count the number of paths of length eight from z to itself.



[2] Solve the following system of equations.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 6 & 8 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 3 \end{bmatrix}$$

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ -4 & 5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

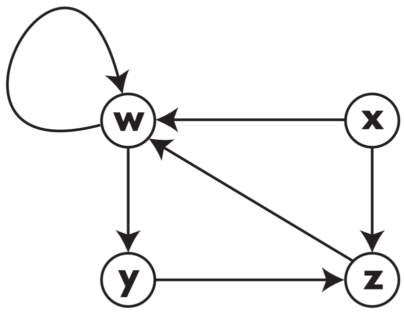
[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

(S13 Alt Exam 1) (Solutions)

[1] Using matrix multiplication, count the number of paths of length nine from y to itself.



[2] Solve the following system of equations.

$$\begin{bmatrix} 2 & 4 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

(S13 8:40 Exam 2) (Solutions)

[1] Find a basis for the set of solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Extend this basis to a basis for $\mathbb{R}^4$.

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (0, 0), \quad (x_2, y_2) = (1, 0), \quad (x_3, y_3) = (3, 1)$$

[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the line

$$x = y = 2z$$

Find the matrix A which represents L in standard coordinates.

[4] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ given by the equation $w + x + y - 2z = 0$.

[5] Let V be the vector space of all polynomials of degree $\leqslant 3$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(0) = f(1) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

(S13 10:10 Exam 2) (Solutions)

[1] Find a basis for the set of solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 2 & 2 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Extend this basis to a basis for $\mathbb{R}^5$.

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (0, 1), \quad (x_2, y_2) = (1, 0), \quad (x_3, y_3) = (2, 2)$$

[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the subspace

$$x + 2y + z = 0$$

Find the matrix A which represents L in standard coordinates.

[4] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,1,1), \quad (1,2,1,2), \quad (2,1,2,1), \quad (2,2,2,2)$$

[5] Let V be the vector space of all polynomials of degree $\leqslant 3$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(0) = f'(0) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

(S13 Alt Exam 2) (Solutions)

[1] Find a basis for the set of solutions to the system of equations

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Extend this basis to a basis for $\mathbb{R}^4$.

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (-1, 0), \quad (x_2, y_2) = (0, 0), \quad (x_3, y_3) = (1, 2)$$

[3] Let L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects first across the line $y = x$, then across the line $y = 3x$. Find the matrix A which represents L in standard coordinates.

[4] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$(1,1,1,1), \quad (1,2,2,2), \quad (1,1,2,2), \quad (3,4,5,5)$$

[5] Let V be the vector space of all polynomials of degree $\leqslant 4$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of odd polynomials: those polynomials for which $f(-x) = -f(x)$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(1-x)\,dx$$

(S12 Practice 1) (Solutions)

[1] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

[2] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

[3] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

[4] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

[6] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

[7] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

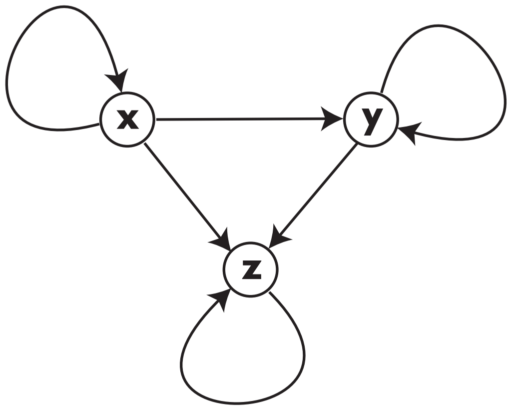
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(S12 9:10 Exam 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 3 & -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length three from x to z.



[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

[4] Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $L(v) = v$ for all v on the line $x + 2y = 0$, and $L(v) = 2v$ for all v on the line $x = y$. Find a matrix A that represents L in standard coordinates.

[5] Find a basis for the subspace V of $\mathbb{R}^4$ given by the equation $w + x + y + 2z = 0$. Extend this basis to a basis for all of $\mathbb{R}^4$.

[6] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

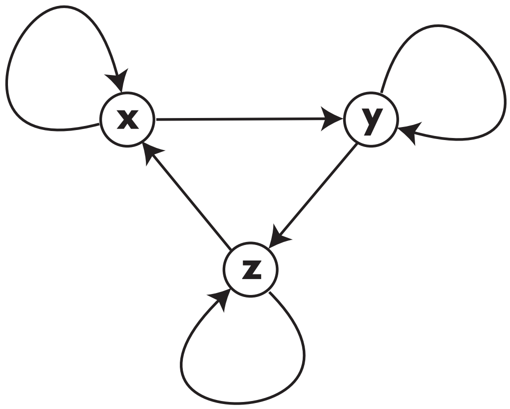
$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(S12 11:00 Exam 1) (Solutions)

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -1 & -2 \\ 2 & 3 & -2 & -3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

[2] Using matrix multiplication, count the number of paths of length three from x to z.



[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}.$$

[4] Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects across the x-axis, and then reflects across the line $y = x$. Find a matrix A that represents L in standard coordinates.

[5] Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the rows of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \end{bmatrix}.$$

Extend this basis to a basis for all of $\mathbb{R}^4$.

[6] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

(S11 Exam 1) (Solutions)

[1] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

[2] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 0 & 1 & 1 & 5 & 0 & 9 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

[3] Use Gaussian elimination to find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

[4] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

[5] Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix which flips the plane $\mathbb{R}^2$ across the line $3x = y$. Find A.

[6] Using matrix multiplication, count the number of paths of length four from y to itself.