Practice Final Exam
Modern Algebra II, Dave Bayer, December 2010

Name: ______________________________________

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Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

[1] The polynomial
\[ g(a, b) = (a - b)^4 \]
is symmetric in \( a \) and \( b \). Express \( g \) as a polynomial in the elementary symmetric functions
\[ s_1 = a + b, \quad s_2 = ab. \]

[2] The polynomial
\[ g(a, b, c) = a^3 + b^3 + c^3 \]
is symmetric in \( a, b, \) and \( c \). Express \( g \) as a polynomial in the elementary symmetric functions
\[ s_1 = a + b + c, \quad s_2 = ab + ac + bc, \quad s_3 = abc. \]

[3] The polynomial
\[ g(a, b, c) = (a - b)^2(a - c)^2(b - c)^2 \]
is symmetric in \( a, b, \) and \( c \). Suppose that
\[ s_1 = a + b + c = 0. \]
Express \( g \) as a polynomial in the remaining elementary symmetric functions
\[ s_2 = ab + ac + bc, \quad s_3 = abc. \]

[4] What is the irreducible polynomial for \( \alpha = \sqrt{2} + \sqrt{3} \) over \( \mathbb{Q} \)?

[5] Let \( f(x) = x^3 - 12 \). What is the degree of the splitting field \( K \) of \( f \) over \( \mathbb{Q} \)?
What is the Galois group \( G = G(K/\mathbb{Q}) \) of \( f \)?
List the subfields \( L \) of \( K \), and the corresponding subgroups \( H = G(K/L) \) of \( G \).

[6] Which of the following cubic polynomials have \( A_3 \) for their Galois group? Which have \( S_3 \) for their Galois group?
\[ x^3 - 21x + 7, \quad x^3 - 3x^2 + 1, \quad x^3 + x^2 + x + 1 \]
[7] Let $K$ be the splitting field over $\mathbb{Q}$ of the polynomial $x^2 - 2x + 3$. Find an element $a \in \mathbb{Q}$ such that $K = \mathbb{Q}(\sqrt{a})$.

[8] Let $K$ be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 + px + q$, where $p, q \in \mathbb{Q}$. When is the degree $[K : \mathbb{Q}] = 3$? When is the degree $[K : \mathbb{Q}] = 6$? Give an example of a polynomial for each case.

[9] Let $K$ be the splitting field over $\mathbb{Q}$ of the polynomial $x^5 - 81x + 3$. What is the Galois group $G(K/\mathbb{Q})$?

[10] Let $F$ be the splitting field of the polynomial $x^p - 1$ over $\mathbb{Q}$, where $p$ is a prime. What is the Galois group $G(F/\mathbb{Q})$?

[11] Let $F$ be the splitting field of the polynomial $x^n - 1$ over $\mathbb{Q}$, where $n$ is a positive integer. What is the Galois group $G(F/\mathbb{Q})$?

[12] Give an example of a degree two polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $C_2$.

[13] Give an example of a degree three polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $A_3$.

[14] Give an example of a degree three polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $S_3$.

[15] Give an example of a degree four polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $C_2 \times C_2$.

[16] Give an example of a degree four polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $C_4$.

[17] Give an example of a degree five polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $S_5$.

[18] Give an example of a degree six polynomial $g(x)$ over $\mathbb{Q}$, whose Galois group is $C_6$.

Proofs

[19] Let $K = F(\alpha, \beta)$ be a finite extension of a field $F$ of characteristic zero. Prove that there is an element $\gamma \in K$ such that $K = F(\gamma)$.

[20] Let $F$ be a subfield of $\mathbb{C}$ that contains all roots of the polynomial $x^p - 1$, where $p$ is a prime. Let $K/F$ be a Galois extension of degree $p$. Prove that $K = F(\sqrt[p]{b})$ for some $b \in F$. 